LANL xxx E-print archive No. quant-ph/0004041

Fibre bundle formulation of nonrelativistic quantum mechanics (full version)

Bozhidar Z. Iliev * † ‡

Short title: **Bundle quantum mechanics**

Basic ideas: → March 1996
Began: → May 19, 1996
Ended: → July 12, 1996
Revised: → December 1996 – January 1997,
Revised: → April 1997, September 1998, June 1999
Last updated: → April 4, 2000
Produced: → February 1, 2008

LANL xxx archive server E-print No.: quant-ph/0004041

BOZHO® TM

Subject Classes:
Quantum mechanics; Differential geometry

1991 MSC numbers:
81P05, 81P99, 81Q99, 81S99

1996 PACS numbers:
02.40.Ma, 04.60.-m, 03.65.Ca, 03.65.Bz

Key-Words:
Quantum mechanics; Geometrization of quantum mechanics; Fibre bundles

*Department Mathematical Modeling, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria
†E-mail address: bozho@inrne.bas.bg
‡URL: http://www.inrne.bas.bg/mathmod/bozhome/

Contents

Distribution of the material:

Part	Sections	Title	quant-ph/#	Ref.
0		Preliminary considerations	9803083	[1]
I	1–5	Introduction. The evolution transport	9803084	[2]
II	6&7	Equations of motion and observables	9804062	[3]
III	8&9	Pictures and integrals of motion	9806046	[4]
IV	10&11	Mixed states and evolution transport's curvature	9901039	[5]
V	12–14	Theory's interpretation, summary and discussion	9902068	[6]

Abstract

We propose a new systematic fibre bundle formulation of nonrelativistic quantum mechanics. The new form of the theory is equivalent to the usual one and is in harmony with the modern trends in theoretical physics and potentially admits new generalizations in different directions. In it the Hilbert space of a quantum system (from conventional quantum mechanics) is replaced with an appropriate Hilbert bundle of states and a pure state of the system is described by a lifting of paths or section along paths in this bundle. The evolution of a pure state is determined through the bundle (analogue of the) Schrödinger equation. Now the dynamical variables and the density operator are described via liftings of paths or morphisms along paths in suitable bundles. The mentioned quantities are connected by a number of relations derived in this work.

1. Introduction

Usually the standard nonrelativistic quantum mechanics of pure states is formulated in terms of vectors and operators in a Hilbert space [7–11]. This is in discrepancy and not in harmony with the new trends in (mathematical) physics [12–14] in which the theory of fibre bundles [15,16], in particular vector bundles [17, 18], is essentially used. This paper (and its further continuation(s)) is intended to incorporate the quantum theory in the family of fundamental physical theories based on the background of fibre bundles.

The idea of geometrization of quantum mechanics is an old one (see, e.g., [19] and the references therein). A good motivation for such approach is given in [19,20]. Different geometrical structures in quantum mechanics were introduced [21, 22], for instance such as inner products(s) [8, 9, 19, 23], (linear) connection [20,23,24], symplectic structure [20], complex structure [19], etc. The introduction of such structures admits a geometrical treatment of some problems, for instance, the dynamics in the (quantum) phase space [19] and the geometrical phase [20]. In a very special case, a gauge structure, i.e. a parallel transport corresponding to a linear connection, in quantum mechanics is pointed out in [25]. For us this work is remarkable with the fact that the equation (10) in it is a very 'ancient' special version of our equation (6.24) (see also below Sect. 6). It, together with equation (6.25), shows that (up to a constant) with respect to the quantum evolution the Hamiltonian plays the rôle of a gauge field (linear connection). In [24, 26] one finds different (vector) bundles defined on the base of the (usual) Hilbert space of quantum mechanics or its modifications. In these works different parallel transports in the corresponding bundles are introduced too. Some interesting ideas concerning the theory of fibre bundles and quantum mechanics can be found in [27]. A recent review of the foundations of geometric quantization can be found in [28].

A general feature of all of the references above-cited is that in them all geometric concepts are introduced by using in one or the other way the accepted mathematical foundation of quantum mechanics, viz. a suitable Hilbert or projective Hilbert space and operators acting in it. The Hilbert space may be extended in a certain sense or replaced by a more general space, but this does not change the main ideas. One of the aims of this work is namely to change this mathematical background of quantum mechanics.

Separately we have to mention the approach of Prugovečki to the quantum theory, a selective summary of which can be found in [29] (see also the references therein) and in [30]. It can be characterized as 'stochastic' and 'bundle'. The former feature will not be discussed in the present investigation; thus we lose some advantages of the stochastic quantum theory to which we shall return elsewhere. The latter 'part' of the Prugovečki's approach has some common aspects with our present work but, generally, it is essentially different. For instance, in both cases the quantum evolution from

point to point (in space-time) is described via a kind of (parallel or generic linear) transport (along paths) in a suitable Hilbert fibre bundle. But the notion of a 'Hilbert bundle' in our and Prugovečki's approach is different regardless that in both cases the typical (standard) fibre is practically the same (when one and the same theory is concerned). Besides, we need not even to introduce the Poincaré (principal) fibre bundle over the space-time or the phase space which play an important rôle in Prugovečki's theory. Also we have to notice that the used in it concepts of quantum and parallel transport are special cases of the notion of a 'linear transport along paths' introduced in [31,32]. The application of the last concept, which is accepted in the present investigation, has a lot of advantages, significantly simplifies some proofs and makes certain results 'evident' or trivial (e.g. the last part of section 2 and the whole section 4 of [29]). At last, at the present level (nonrelativistic quantum mechanics) our bundle formulation of the quantum theory is insensitive with respect to the space-time curvature. A detail comparison of Prugovečki's and our approaches to the quantum theory will be done elsewhere.

Another geometric approach to quantum mechanics is proposed in [33] and partially in [34], the letter of which is, with a few exceptions, almost a review of the former. These works suggest two ideas which are quite important for us. First, the quantum evolution could be described as a (kind of) parallel transport in an infinitely dimensional (Hilbert) fibre bundle over the space-time. And second, the concrete description of a quantum system should explicitly depend on (the state of) the observer with respect to which it is depicted (or who 'investigates' it). These ideas are incorporated and developed in our work.

From the known to the author literature, the work is [35] closest to the approach developed here. It contains an excellent motivation for applying the fibre bundle technique to nonrelativistic quantum mechanics.[1] Generally said, in this paper the evolution of a quantum system is described as a 'generalized parallel transport' of appropriate objects in a Hilbert fibre bundle over the 1-dimensional manifold $\mathbb{R}_+ := \{t : t \in \mathbb{R}, t \geq 0\}$, interpreted as a 'time' manifold (space). We shall comment on reference [35] later, after developing the formalism required for its analysis (see below Sect. 6). Besides, we emphasizes once again, the paper [35] contains an excellent motivation why the apparatus of fibre bundle theory is a natural scene for quantum mechanics.

An attempt to formulate quantum mechanics in terms of a fibre bundle over the phase space is made in [36]. Regardless of some common features, this paper is quite different from the present investigation on which we shall comment later (see Sect. 14). In particular, in [36] the gauge structure of

[1] The author thanks J.F. Coriñena (University of Zaragoza, Zaragoza, Spain) for drawing his attention to reference [35] in May 1998.

the arising theory is governed by a non-dynamical connection related to the symplectic structure of the system's phase space, while in this work analogous structure (linear transport along paths) is uniquely connected with system's Hamiltonian, playing here the rôle of a gauge field itself.

The present work is a direct continuation of the considerations in [1] which paper, in fact, may be regarded as its preliminary version. Here we suggest a *purely fibre bundle formulation of the nonrelativistic quantum mechanics*. The proposed geometric formulation of quantum mechanics is *dynamical* in a sense that all geometrical structures employed for the description of a quantum system depend on and are determined form the dynamical characteristics of the system. This new form of the theory is *entirely equivalent* to the usual one, which is a consequence or our step by step equivalent reformulation of the quantum theory. The bundle description is obtained on the base of the developed by the author theory of transports along paths in fibre bundles [31, 32, 37], generalizing the theory of parallel transport. It is partially generalized here to the infinitely dimensional case.

The main object in quantum mechanics is the Hamiltonian (operator) which, through the Schrödinger equation, governs the evolution of a quantum system [8–11]. In our novel approach its rôle is played by a suitable linear transport along paths in an appropriate Hilbert bundle. It turns out that up to a constant the matrix-bundle Hamiltonian, which is uniquely determined by the Hamiltonian in a given field of bases, coincides with the matrix of the coefficients of this transport (cf. an analogous result in [1, sect. 5]). This fact, together with the replacement of the usual Hilbert space with a Hilbert bundle, is the corner-stone for the possibility for the new formulation of the nonrelativistic quantum mechanics.

The present paper is organized as follows.

In Sect. 2 are reviewed some well known facts from the quantum standard mechanics and partially is fixed our notation. Here, as well as throughout this work, we follow the established in the physical literature degree of rigor. But, if required, the present work can be reformulated to meet the present-day mathematical standards. For this purpose one can use, for instance, the quantum-mechanical formalism described in [10] or in [38] (see also [39, 40]).

Sect. 3 contains a preliminary mathematical material required for the goals of this work. In Subsect. 3.1 are collected some basic definitions concerning vector and Hilbert bundles. Next, in Subsect. 3.2, the notion of a linear transport along paths in vector bundles is recalled and some its peculiarities in the Hilbert bundle case are pointed. In Subsect. 3.3 the concepts of liftings of paths, sections along paths and derivations along paths are introduced.

Sect. 4 begins the building of the new bundle approach to quantum mechanics. After a brief review of some references (Subsect. 4.1), a motivation

for the application of fibre bundle formalism to quantum mechanics is presented (Subsect. 4.2). Also some heuristic considerations of elements of the new theory are given. Subsect. 4.3 introduces the basic initial assertions of the bundle version of quantum mechanics. They are formulated in a form of two postulates which are enough for the bundle description of the evolution of a quantum system. In the bundle approach the analogues of the Hilbert space of states and state vector of a system are the system's Hilbert bundle (of states) and the (state) lifting of paths (or section along paths) in it. In Subsect. 4.4 a preliminary summary of some results of this investigation is presented.

In Sect. 5 is proved that in the bundle description the evolution operator of a quantum system is (equivalently) replaced by a suitable linear transport along paths, called *evolution transport*.

Sect. 6 is devoted to the bundle analogues of the Schrödinger equation which are fully equivalent to it. In particular, in it is introduced the *matrix-bundle Hamiltonian* which governs the quantum evolution through the *matrix-bundle Schrödinger equation*. The corresponding matrix of the bundle-evolution transport is found. It is proved that *up to a constant the matrix of the coefficients of the bundle evolution transport coincides with the matrix-bundle Hamiltonian*. On this basis is derived the *(invariant) bundle-Schrödinger equation*. Geometrically it simply means that the corresponding state sections are (parallelly, or, more precisely, linearly) transported by means of the bundle evolution transport (along paths).

In Sect. 7 is considered the question for the bundle description of observables. It turns out that *to any observable there corresponds a unique Hermitian bundle morphism (along paths) and vice versa*.

The bundle description of the different pictures of motion is presented in Sect. 8. The Schrödinger picture, which, in fact, is investigated until Sect. 8, is reviewed in Subsection 8.1. To the bundle Heisenberg picture is devoted Subsection 8.2. The corresponding equations of motion for the observables are derived and discussed. In Subsection 8.3 is investigated the 'general' picture of motion obtained by means of an arbitrary linear unitary bundle map (along paths). In it are derived and discussed different equations for the state sections and observables.

In Sect. 9 are investigated problems concerning the integrals of motion from fibre bundle point of view. An interesting result here is that a dynamical variable is an integral of motion iff the corresponding to it bundle morphism is transported along the observer's world line with the help of the (bundle) evolution transport.

The bundle approach to the quantum mechanics of mixed states is investigated in Sect. 10. Subsection 10.1 is a brief review of the conventional concepts of mixed state(s) and density operator (matrix). Their bundle description is presented in Subsection 10.2. It turns out that to the density operator there corresponds a suitable *density morphism along paths*. The

equations for its time evolution are derived. In Subsection 10.4 are studied problems connected with the representations and description of mixed quantum states in the different pictures of motion. The equations of motion for density morphisms and operators are derived.

Sect. 11 is devoted to the curvature of the (bundle) evolution transport.

In Sect. 12 is paid attention on the observer's rôle in the theory and are considered and interpreted some modifications of the proposed approach to quantum mechanics. Possible fields for further research are sketched too.

In Sect. 13 are briefly summarized our results and it is presented a comparison table between the conventional, Hilbert space, and the new, Hilbert bundle, formulations of nonrelativistic quantum mechanics.

In Sect. 14 are discussed certain aspects of the bundle formulation of nonrelativistic quantum mechanics and are pointed some its possible generalizations and applications.

2. Evolution of pure quantum states (review)

In quantum mechanics [8, 9, 11, 40] a pure state of a quantum system is described by a state vector $\psi(t)$ (in Dirac's [7] notation $|t\rangle$). Generally, depends on the time $t \in \mathbb{R}$ and belongs to a Hilbert space $\mathcal{F}$ (specific to any concrete system) generically endowed with a nondegenerate Hermitian scalar product $\langle\cdot|\cdot\rangle\colon \mathcal{F}\times\mathcal{F}\to\mathbb{C}$.[2] For every two instants of time t_2 and t_1 the corresponding state vectors are connected by the equality

$$\psi(t_2) = \mathcal{U}(t_2,t_1)\psi(t_1) \tag{2.1}$$

where $\mathcal{U}$ is the *evolution operator* of the system [10, chapter IV, Sect. 3.2]. It is supposed to be linear and unitary, i.e.

$$\mathcal{U}(t_2,t_1)(\lambda\psi(t_1)+\mu\xi(t_1)) = \lambda\mathcal{U}(t_2,t_1)(\psi(t_1)) + \mu\mathcal{U}(t_2,t_1)(\xi(t_1)), \tag{2.2}$$

$$\mathcal{U}^\dagger(t_1,t_2) = \mathcal{U}^{-1}(t_2,t_1) \tag{2.3}$$

for every $\lambda,\mu\in\mathbb{C}$ and state vectors $\psi(t),\xi(t)\in\mathcal{F}$, and such that for any t

$$\mathcal{U}(t,t) = \mathsf{id}_{\mathcal{F}}. \tag{2.4}$$

Here id_X means the identity map of a set X and the dagger ($\dagger$) denotes Hermitian conjugation, i.e. if $\varphi,\psi\in\mathcal{F}$ and $\mathcal{A}\colon\mathcal{F}\to\mathcal{F}$, then $\mathcal{A}^\dagger$ is defined by

$$\langle\mathcal{A}^\dagger\varphi|\psi\rangle = \langle\varphi|\mathcal{A}\psi\rangle. \tag{2.5}$$

[2] For some (e.g. unbounded) states the system's state vectors form a more general space than a Hilbert one. This is insignificant for the following presentation. A sufficient for our purposes summary of Hilbert space theory is given in [41, Appendix].

In particular $\mathcal{U}^\dagger$ is defined by $\langle \mathcal{U}^\dagger(t_1,t_2)\varphi(t_2)|\psi(t_1)\rangle = \langle \varphi(t_2)|\mathcal{U}(t_2,t_1)\psi(t_1)\rangle$. So, knowing $\psi(t_0)=\psi_0$ for some moment t_0, one knows the state vector for every moment t as $\psi(t)=\mathcal{U}(t,t_0)\psi(t_0)=\mathcal{U}(t,t_0)\psi_0$.

Let $\mathcal{H}(t)$ be the Hamiltonian (function) of the system. It, generally, depends on the time t explicitly[3] and it is supposed to be a Hermitian operator, i.e. $\mathcal{H}^\dagger(t)=\mathcal{H}(t)$. The Schrödinger equation (see [7, § 27] or [10, chapter V, Sec. 3.1])

$$\mathrm{i}\hbar\frac{d\psi(t)}{dt}=\mathcal{H}(t)\psi(t), \tag{2.6}$$

with $\mathrm{i}\in\mathbb{C}$ and $\hbar$ being respectively the imaginary unit and the Plank's constant (divided by 2π), together with some initial condition

$$\psi(t_0)=\psi_0\in\mathcal{F} \tag{2.7}$$

is postulated in the quantum mechanics. They determine the time-evolution of the state vector $\psi(t)$.

The substitution of (2.1) into (2.6) shows that there is a bijective correspondence between $\mathcal{U}$ and $\mathcal{H}$ described by

$$\mathrm{i}\hbar\frac{\partial\mathcal{U}(t,t_0)}{\partial t}=\mathcal{H}(t)\circ\mathcal{U}(t,t_0),\qquad \mathcal{U}(t_0,t_0)=\mathsf{id}_{\mathcal{F}} \tag{2.8}$$

where $\circ$ denotes composition of maps. If $\mathcal{U}$ is given, then

$$\mathcal{H}(t)=\mathrm{i}\hbar\frac{\partial\mathcal{U}(t,t_0)}{\partial t}\circ\mathcal{U}^{-1}(t,t_0)=\mathrm{i}\hbar\frac{\partial\mathcal{U}(t,t_0)}{\partial t}\circ\mathcal{U}(t_0,t), \tag{2.9}$$

where we have used the equality

$$\mathcal{U}^{-1}(t_2,t_1)=\mathcal{U}(t_1,t_2)$$

which follows from (2.1) (see also below (2.10) or Sect. 5). Conversely, if $\mathcal{H}$ is given, then [9, chapter VIII, § 8] $\mathcal{U}$ is the unique solution of the integral equation $\mathcal{U}(t,t_0)=\mathsf{id}_{\mathcal{F}}+\frac{1}{\mathrm{i}\hbar}\int\limits_{t_0}^{t}\mathcal{H}(\tau)\mathcal{U}(\tau,t_0)d\tau$, i.e. we have

$$\mathcal{U}(t,t_0)=\mathrm{Texp}\int\limits_{t_0}^{t}\frac{1}{\mathrm{i}\hbar}\mathcal{H}(\tau)d\tau, \tag{2.10}$$

[3]Of course, the Hamiltonian depends also on the observer with respect to which the evolution of the quantum system is described. This dependence is usually implicitly assumed and not written explicitly [7, 9]. This deficiency will be eliminated in a natural way further in the present work. The Hamiltonian can also depend on other quantities, such as the (operators of the) system's generalized coordinates. This possible dependence is insignificant for our investigation and will not be written explicitly.

where $\operatorname{Texp}\int_{t_0}^{t}\cdots d\tau$ is the chronological (called also T-ordered, P-ordered or path-ordered) exponent (defined, e.g., as the unique solution of the initial-value problem (2.8); see also [35, equation (1.3)]).[4] From here follows that the Hermiticity of $\mathcal{H}$, $\mathcal{H}^\dagger = \mathcal{H}$, is equivalent to the unitarity of $\mathcal{U}$ (see (2.3)).

Let us note that for the rigorous mathematical understanding of the derivations in (2.6), (2.8), and (2.9), as well as of the chronological (path-ordered) exponent in (2.10), one has to apply the developed in [10] mathematical apparatus, but this is out of the subject of the present work.

If $\mathcal{A}(t)\colon \mathcal{F} \to \mathcal{F}$ is the (linear Hermitian) operator corresponding to a dynamical variable $\mathbb{A}$ at the moment t, then the mean value (= the mathematical expectation) which it assumes at a state described by a state vector $\psi(t)$ with a finite norm is

$$\langle \mathcal{A} \rangle_\psi^t := \langle \mathcal{A}(t) \rangle_\psi^t := \langle \mathcal{A}(t) \rangle_{\psi(t)} := \frac{\langle \psi(t) | \mathcal{A}(t)\psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}. \tag{2.11}$$

It is interpreted as the observed value of $\mathbb{A}$ that can be measured experimentally.

Often the operator $\mathcal{A}$ can be chosen independent of the time t. (This is possible, e.g., if $\mathcal{A}$ does not depend on t explicitly [9, chapter VII, § 9] or if the spectrum of $\mathcal{A}$ does not change in time [8, chapter III, sect. 13].) If this is the case, it is said that the system's evolution is depicted in the Schrödinger picture of motion [7, § 28], [9, chapter VII, § 9].

3. Mathematical preliminaries

Before starting with the formulation of quantum mechanics is terms of fibre bundles, several geometrical tools have to be known. In this section is collected most of the pure mathematical material required for this goal. At first, we present some facts from the theory of Hilbert bundles. These bundles will replace the Hilbert spaces in quantum mechanics. Next, we give a brief introduction to the theory of linear transports along paths in vector bundles and specify peculiarities of the Hilbert bundle case. The linear transports are needed for the bundle description of the quantum evolution. At the end, we pay attention to the liftings of paths and section along paths. These objects will replace the state vectors of ordinary quantum mechanics.

3.1. Hilbert bundles

At the beginning, to fix the terminology, we recall the definitions of bundle, section, fibre map, morphism, and vector bundle.[5]

[4] The physical meaning of $\mathcal{U}$ as a propagation function, as well as its explicit calculation (in component form) via $\mathcal{H}$ can be found, e.g., in [42, § 21 and § 22]

[5] For details, examples, etc., see [15, 17, 18, 43–46].

A *bundle* is a triple (E, π, B) of sets E and B, called (total) bundle space and base (space) respectively, and (generally) surjective mapping $\pi\colon E \to B$, called projection. For every $b \in B$ the set $\pi^{-1}(b)$ is called fibre over b. If $X \subseteq B$, the bundle $(E, \pi, B)|_X := (\pi^{-1}(X), \pi|_{\pi^{-1}(X)}, X)$ is called the restriction on X of a bundle (E, π, B). A *section* of the bundle (E, π, B) is a mapping $\sigma\colon B \to E$ such that $\pi \circ \sigma = \mathsf{id}_B$, i.e. $\sigma\colon b \mapsto \sigma(b) \in \pi^{-1}(b)$. The set of sections of (E, π, B) is denoted by $\mathrm{Sec}(E, \pi, B)$.

A mapping $\varphi\colon E \to E$ is said to be a *fibre map* if it carries fibres into fibres. Precisely, φ is a fibre map iff, for every $b \in B$, there exists a point $b' \in B$ such that φ maps $\pi^{-1}(b)$ into $\pi^{-1}(b')$, i.e. $\varphi|_{\pi^{-1}(b)}\colon \pi^{-1}(b) \to \pi^{-1}(b')$. A *(fibre) morphism* of the bundle (E, π, B) is a pair (φ, f) of maps $\varphi\colon E \to E$ and $f\colon B \to B$ such that $\pi \circ \varphi = f \circ \pi$. The set of morphisms of (E, π, B) is denoted by $\mathrm{Mor}(E, \pi, B)$, i.e.

$$\mathrm{Mor}(E, \pi, B) := \{(\varphi, f) | \varphi\colon E \to E,\ f\colon B \to B,\ \pi \circ \varphi = f \circ \pi\}.$$

A map $\varphi\colon E \to E$ is called *morphism over* B or B*-morphism* if $(\varphi, \mathsf{id}_B) \in \mathrm{Mor}(E, \pi, B)$. The set of all B-morphisms of (E, π, B) will be denoted by $\mathrm{Mor}_B(E, \pi, B)$, i.e.

$$\mathrm{Mor}_B(E, \pi, B) := \{\varphi | \varphi\colon E \to E,\ \pi \circ \varphi = \pi\}.$$

For every morphism $(\varphi, f) \in \mathrm{Mor}(E, \pi, B)$, the map φ is a fibre map since from $\pi \circ \varphi = f \circ \pi$ follows $\varphi|_{\pi^{-1}(b)}\colon \pi^{-1}(b) \to \pi^{-1}(f(b))$ for every $b \in B$. In particular, any B-morphism $\varphi \in \mathrm{Mor}_B(E, \pi, B)$ is a fibre-preserving map as $\varphi|_{\pi^{-1}(b)}\colon \pi^{-1}(b) \to \pi^{-1}(b)$. Conversely, every fibre map $\varphi\colon E \to E$ defines a morphism (φ, f) with $f := \pi \circ \varphi \circ \pi^{-1}\colon B \to B$; f is called *induced map* of the fibre map φ, resp. (φ, f) is *induced morphism*.[6]

Consider the set of point-restricted morphisms

$$\begin{aligned} E_0 :&= \{(\varphi_b, f)\,|\,\varphi_b = \varphi|_{\pi^{-1}(b)},\ b \in B,\ (\varphi, f) \in \mathrm{Mor}(E, \pi, B)\} \\ &= \{(\varphi_b, f)\,|\,\varphi_b\colon \pi^{-1}(b) \to \pi^{-1}(f(b)),\ b \in B,\ f\colon B \to B\}, \end{aligned}$$

i.e. $(\psi, f) \in E_0$ iff $f\colon B \to B$ and there exists a unique $b \in B$ such that $\psi\colon \pi^{-1}(b) \to \pi^{-1}(f(b))$ and we write ψ_b for ψ. Defining $\pi_0\colon E_0 \to B$ by $\pi_0(\varphi_b, f) := b$ for $(\varphi_b, f) \in E_0$, we see that (E_0, π_0, B) is a bundle over the same base B as (E, π, B). This is the *bundle of point-restricted morphisms* of (E, π, B). It will be denoted by $\mathrm{mor}(E, \pi, B)$, i.e. $\mathrm{mor}(E, \pi, B) := (E_0, \pi_0, B)$. There exists a bijective correspondence τ such that

$$\mathrm{Mor}(E, \pi, B) \xrightarrow{\tau} \mathrm{Sec}\big(\mathrm{mor}(E, \pi, B)\big).$$

In fact, for $(\varphi, f) \in \mathrm{Mor}(E, \pi, B)$, we put $\tau\colon (\varphi, f) \mapsto \tau_{(\varphi,f)}$ with $\tau_{(\varphi,f)}\colon b \mapsto \tau_{(\varphi,f)}(b) := (\varphi|_{\pi^{-1}(b)}, f) \in \pi_0^{-1}(b)$ for every $b \in B$. Conversely, for $\sigma \in$

[6] The transport along a path $\gamma\colon J \to B$ is an example of a fibre map along γ – *vide infra* Subsect. 3.2.

$\mathrm{Sec}(\mathrm{mor}(E,\pi,B))$, we set $\tau^{-1}\colon \sigma \mapsto \tau^{-1}(\sigma) := (\varphi, f) \in \mathrm{Mor}(E,\pi,B)$ where, if $b \in B$ and $\sigma(b) = (\varphi_b, f)$, the map $\varphi\colon E \to E$ is defined by $\varphi|_{\pi^{-1}(b)} := \varphi_b$.

The above constructions can be modified for morphisms over the bundle's base as follows. The *bundle* $\mathrm{mor}_B(E,\pi,B)$ *of point-restricted morphisms over* B of (E,π,B) has a base B, bundle space

$$\begin{aligned} E_0^B &:= \{\varphi_b \,|\, \varphi_b = \varphi|_{\pi^{-1}(b)},\ b \in B,\ \varphi \in \mathrm{Mor}_B(E,\pi,B)\} \\ &= \{\varphi_b \,|\, \varphi_b\colon \pi^{-1}(b) \to \pi^{-1}(b),\ b \in B\} \end{aligned}$$

and projection $\pi_0^B\colon E_0^B \to B$ such that

$$\pi_0^B(\varphi_b) := b, \quad \varphi_b \in E_0^B.$$

For brevity, the bundle $\mathrm{mor}_B(E,\pi,B)$ will be refereed as the *bundle of restricted morphisms* of (E,π,B). Evidently, the set E_0^B coincides with the set of point-restricted fibre-preserving fibre maps of (E,π,B). There is a bijection

$$\mathrm{Mor}_B(E,\pi,B) \xrightarrow{\chi} \mathrm{Sec}\big(\mathrm{mor}_B(E,\pi,B)\big)$$

given by $\chi\colon \varphi \mapsto \chi_\varphi$, $\varphi \in \mathrm{Mor}_B(E,\pi,B)$, with $\chi_\varphi\colon b \mapsto \chi_\varphi(b) := \varphi|_{\pi^{-1}(b)}$, $b \in B$. Its inverse is $\chi^{-1}\colon \sigma \mapsto \chi^{-1}(\sigma) = \varphi$, $\sigma \in \mathrm{Sec}(\mathrm{mor}_B(E,\pi,B))$, with $\varphi\colon E \to E$ given via $\varphi|_{\pi^{-1}(b)} = \sigma(b)$ for every $b \in B$.

If E and B are topological spaces, which is the most widely considered case, the bundle (E,π,B) is called *topological*. In this case in the definition of a bundle is included the *bundle property*: there exists a (topological) space $\mathcal{E}$ such that for each $b \in B$, there is an open neighborhood ('directory space') W of b in B and homeomorphism ('decomposition function') $\phi_W\colon W \times \mathcal{E} \to \pi^{-1}(W)$ of $W \times \mathcal{E}$ onto $\pi^{-1}(W)$ satisfying the condition $(\pi \circ \phi_W)(w,e) = w$ for $\psi \in W$ and $e \in \mathcal{E}$. Besides, if the restriction $\phi_W|_b\colon \{b\} \times \mathcal{E} \to \pi^{-1}(b)$, $b \in B$, is homeomorphism, the bundle property is called *local triviality*, $\mathcal{E}$ is called *(typical, standard) fibre of the bundle*, and every fibre $\pi^{-1}(b)$ is homeomorphic to $\mathcal{E}$ for every $b \in B$.

A *vector bundle* is locally trivial bundle (E,π,B) such that: (i) the fibres $\pi^{-1}(b)$, $b \in B$ and the standard fibre $\mathcal{E}$ are (linearly) isomorphic vector spaces and (ii) the decomposition mappings ϕ_W and their restrictions $\phi_W|_b$ are (linear) isomorphisms between vector spaces. The dimension of $\mathcal{E}$, $\dim \mathcal{E} = \dim \pi^{-1}(b)$ for every $b \in B$, is called dimension of the vector bundle, resp. it is called $\dim \mathcal{E}$-dimensional. Here the vector spaces are considered over some field, usually the real or complex numbers; in the context of the present work, the complex case will be employed.

When vector bundles are considered, in the definition of a morphism or B-morphism is included the condition the corresponding fibre maps to be linear. For example, $\varphi\colon E \to E$ is morphism over B of a vector bundle (E,π,B) if $\pi \circ \varphi = \pi$ and the restricted mapping $\varphi|_{\pi^{-1}(b)}\colon \pi^{-1}(b) \to \pi^{-1}(b)$ is linear for every $b \in B$.

Definition 3.1. A Hilbert (fibre) bundle is a vector bundle whose fibres over the base are isomorphic Hilbert spaces or, equivalently, whose (standard) fibre is a Hilbert space.

In the present investigation we shall show that the Hilbert bundles can be taken as a natural mathematical framework for a geometrical formulation of quantum mechanics.

Some quite general aspects of the Hilbert bundles can be found in [41, chapter VII]. Below we are going to consider only certain specific properties and structures of the Hilbert bundle theory required for the present investigation.

Let (F, π, M) be a Hilbert bundle with bundle space F, base M, projection π, and (typical) fibre $\mathcal{F}$. The fibre over $x \in M$ will be often denoted by F_x, $F_x := \pi^{-1}(x)$. Let $l_x \colon F_x \to \mathcal{F}$, $x \in M$, be the isomorphisms defined by the restricted decomposition functions, viz., as $\phi_W|_x \colon \{x\} \times \mathcal{F} \to F_x$, we define l_x via $\phi_W|_x(x, \psi) =: l_x^{-1}(\psi) \in \pi^{-1}(x)$ for every $\psi \in \mathcal{F}$. We call the maps l_x point-trivializing maps (isomorphisms).

Let $\langle \cdot | \cdot \rangle \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be the (non-degenerate Hermitian) scalar product in the Hilbert space $\mathcal{F}$ and, respectively, for every $x \in M$ the map $\langle \cdot | \cdot \rangle_x \colon F_x \times F_x \to \mathbb{R}$ be the scalar product in the fibre F_x considered as a Hilbert space.[7] For a general Hilbert bundle (F, π, M), the scalar products $\langle \cdot | \cdot \rangle_x$, $x \in M$, and $\langle \cdot | \cdot \rangle$ are completely independent. Such a situation is unsatisfactory from the viewpoint of many applications for which the Hilbert spaces F_x, $x \in M$ and $\mathcal{F}$ is required to be *isometric*. We say that the vector structure of the Hilbert bundle (F, π, M) is *compatible* with its metric structure if the (linear) isomorphisms $l_x \colon F \to \mathcal{F}$ preserve the scalar products (are metric-preserving), viz. iff $\langle \varphi_x | \psi_x \rangle_x = \langle l_x(\varphi_x) | l_x(\psi_x) \rangle$ for every $\varphi_x, \psi_x \in F_x$. A Hilbert bundle with compatible vector and metric structure will be called *compatible Hilbert bundle*. In such a bundle the linear isomorphisms $l_x | x \in M$ not only (isomorphicly) connect the vector structures of the fibres F_x, $x \in M$ and $\mathcal{F}$, but they also transform the (Hermitian) metric structure $\langle \cdot | \cdot \rangle$ from $\mathcal{F}$ to F for every $x \in M$ according to

$$\langle \cdot | \cdot \rangle_x = \langle l_x \cdot | l_x \cdot \rangle, \qquad x \in M \tag{3.1}$$

and, consequently, from F_x to $\mathcal{F}$ through

$$\langle \cdot | \cdot \rangle = \langle l_x^{-1} \cdot | l_x^{-1} \cdot \rangle_x, \qquad x \in M. \tag{3.1^prime}$$

It is easy to see that the maps $l_{x \to y} := l_y^{-1} \circ l_x \colon \pi^{-1}(x) \to \pi^{-1}(y)$ are (i) fibre maps for fixed y, (ii) linear isomorphisms, and (iii) isometric, i.e. metric preserving in a sense that

$$\langle l_{x \to y} \cdot | l_{x \to y} \cdot \rangle_y = \langle \cdot | \cdot \rangle_x. \tag{3.2}$$

[7] The map $x \mapsto \langle \cdot | \cdot \rangle_x$, $x \in M$, or the collection of maps $\{\langle \cdot | \cdot \rangle_x,\ x \in M\}$ is called a *fibre metric* on (F, π, M).

Rewording, we can say that $l_{x\to y}$ are fibre-isometric isomorphisms. Consequently, all of the fibres over the base and the standard fibre of a compatible Hilbert bundle are (linearly) isometric and isomorphic Hilbert spaces.

Beginning from now on in the present investigation, *only compatible Hilbert bundles will be employed*. To save writing and for the sake of brevity, we shall call them simply Hilbert bundles.

Now some definitions in compatible Hilbert bundles are in order. Notice, below we present the minimum of material concerning Hilbert bundles which is absolutely required for formulation of quantum mechanics in terms of bundles.

Defining the *Hermitian conjugate* map (operator) $A_x^\ddagger\colon \mathcal{F}\to F_x$ of a map $A_x\colon F_x\to\mathcal{F}$ by

$$\langle A_x^\ddagger\varphi|\chi_x\rangle_x := \langle\varphi|A_x\chi_x\rangle,\qquad \varphi\in\mathcal{F},\quad \chi_x\in F_x, \tag{3.3}$$

we find (see (3.1))

$$A_x^\ddagger = l_x^{-1}\circ\left(A_x\circ l_x^{-1}\right)^\dagger \tag{3.4}$$

where the dagger denotes Hermitian conjugation in $\mathcal{F}$ (see (2.5)).

We call a map $A_x\colon F_x\to\mathcal{F}$ *unitary* if

$$A_x^\ddagger = A_x^{-1}. \tag{3.5}$$

Evidently, the isometric isomorphisms $l_x\colon F_x\to\mathcal{F}$ are unitary in this sense:

$$l_x^\ddagger = l_x^{-1}. \tag{3.6}$$

Similarly, the *Hermitian conjugate* map to a map $A_{x\to y}\in\{C_{x\to y}\colon F_x\to F_y,\ x,y\in M\}$ is a map $A_{x\to y}^\ddagger\colon F_x\to F_y$ defined via

$$\langle A_{x\to y}^\ddagger\Phi_x|\Psi_y\rangle_y := \langle\Phi_x|A_{y\to x}\Psi_y\rangle_x,\qquad \Phi_x\in F_x,\quad \Psi_y\in F_y. \tag{3.7}$$

Its explicit form is

$$A_{x\to y}^\ddagger = l_y^{-1}\circ\left(l_x\circ A_{y\to x}\circ l_y^{-1}\right)^\dagger\circ l_x. \tag{3.8}$$

As $(\mathcal{A}^\dagger)^\dagger\equiv\mathcal{A}$ for any $\mathcal{A}\colon\mathcal{F}\to\mathcal{F}$, we have

$$\left(A_{x\to y}^\ddagger\right)^\ddagger = A_{x\to y}. \tag{3.9}$$

If $B_{x\to y}\in\{C_{x\to y}\colon F_x\to F_y,\ x,y\in M\}$, then a simple verification shows

$$(B_{y\to z}\circ A_{x\to y})^\ddagger = A_{y\to z}^\ddagger\circ B_{x\to y}^\ddagger,\qquad x,y,z\in M. \tag{3.10}$$

A map $A_{x\to y}$ is called *Hermitian* if

$$A^{\ddagger}_{x\to y} = A_{x\to y}. \tag{3.11}$$

A simple calculation proves that the maps $l_{x\to y} := l_y^{-1} \circ l_x$ are Hermitian.

A map $A_{x\to y}\colon F_x \to F_y$ is called *unitary* if it has a left inverse map and

$$A^{\ddagger}_{x\to y} = A^{-1}_{y\to x}, \tag{3.12}$$

where $A^{-1}_{x\to y}\colon F_y \to F_x$ is the *left* inverse of $A_{x\to y}$, i.e. $A^{-1}_{x\to y} \circ A_{x\to y} := \mathsf{id}_{F_x}$.

A simple verification by means of (3.7) shows the equivalence of (3.12) with

$$\langle A_{y\to x} \cdot | A_{y\to x} \cdot \rangle_x = \langle \cdot | \cdot \rangle_y \colon F_y \times F_y \to \mathbb{C}, \tag{3.12$'$}$$

i.e. the unitary maps are fibre-metric compatible in a sense that they preserve the fibre scalar (inner) product. Such maps will be called *fibre-isometric* or simply *isometric*.

It is almost evident that the maps $l_{x\to y} = l_y^{-1} \circ l_x$ are unitary, that is we have:[8]

$$l^{\ddagger}_{x\to y} = l_{x\to y} = l^{-1}_{y\to x}, \qquad l_{x\to y} := l_y^{-1} \circ l_x \colon \pi^{-1}(x) \to \pi^{-1}(y). \tag{3.13}$$

Let A be a morphism over M of (F, π, M), i.e. $A\colon F \to F$ and $\pi \circ A = \pi$, and $A_x := A|_{F_x}$. The *Hermitian conjugate* bundle morphism $A^{\ddagger}$ to A is defined by (cf. (3.7))

$$\langle A^{\ddagger}\Phi_x | \Psi_x \rangle_x := \langle \Phi_x | A\Psi_x \rangle_x, \qquad \Phi_x, \Psi_x \in F_x. \tag{3.14}$$

Thus (cf. (3.8))

$$A^{\ddagger}_x := A^{\ddagger}\big|_{F_x} = l_x^{-1} \circ \big(l_x \circ A_x \circ l_x^{-1}\big)^{\dagger} \circ l_x. \tag{3.15}$$

A bundle morphism A is called *Hermitian* if $A^{\ddagger}_x = A_x$ for every $x \in M$, i.e. if

$$A^{\ddagger} = A, \tag{3.16}$$

and it is called *unitary* if $A^{\ddagger}_x = A_x^{-1}$ for every $x \in M$, i.e. if

$$A^{\ddagger} = A^{-1}. \tag{3.17}$$

[8]The Hermiticity and at the same time unitarity of $l_{x\to y}$ is not incidental as they define a (flat) linear transport (along paths or along the identity map of M) in (F, π, M) (see (3.23), the paragraph after (3.28), and footnote 11).

Using (3.14), we can establish the equivalence of (3.17) and

$$\langle A \cdot |A\cdot\rangle_x = \langle\cdot|\cdot\rangle_x \colon F_x \times F_x \to \mathbb{C}. \tag{3.17$'$}$$

Consequently the unitary morphisms are fibre-metric compatible, i.e. they are *isometric* in a sense that they preserve the fibre Hermitian scalar (inner) product.

Starting with Sect. 6 of the present work, we will need to deal with the differentiable properties of the employed Hilbert bundle (F, π, M). To make this possible, we will require the (total) bundle space F to be (at least) C^1 manifold.[9] Besides, we shall need the paths in M to have a continuous tangent vectors; in our interpretation of M as a spacetime model, this corresponds to the existence of velocities of the (point-like) particles and observers. To ensure this natural requirement, we assume M to be a C^1 differentiable manifold.[10] Moreover, we shall need the point-trivializing isomorphisms l_x to have a C^1 dependence of $x \in M$, i.e. the mapping $l \colon F \to \mathcal{F}$ given by $l \colon u \mapsto l_{\pi(u)}u$ for $u \in F$, to be of class C^1 as a map between manifolds. A Hilbert bundle with the last property will be called C^1 bundle (or bundle of class C^1).

Let us summarize the basic requirements for the bundle (F, π, M) that will be employed in this work: (i) it is a compatible Hilbert bundle; (ii) the bundle space F and the base M are C^1 differentiable manifolds; and (iii) it is of class C^1 and the set $\{l_x,\ x \in M\}$ of point trivializing (C^1 isometric) isomorphisms is fixed.[11] The isomorphisms l_x will frequently and explicitly be used throughout the present work. The formalism of the theory is not invariant under their choice but the corresponding transformation formulae are easily derivable and the physical predictions are independent of them. For instance, if $\{m_x\}$ is another set of point-trivializing isomorphisms, the scalar products $\langle\cdot|\cdot\rangle^l_x$ and $\langle\cdot|\cdot\rangle^m_x$ defined respectively by $\{l_x\}$ and $\{m_x\}$ are connected via the equality $\langle\cdot|\cdot\rangle^l_x = \langle\varphi_{l,m} \cdot |\varphi_{l,m}\cdot\rangle^m_x$ where the (fibre-preserving) bundle morphism $\varphi_{l,m} \colon F \to F$ is given by $\varphi_{l,m}|_{F_x} := \varphi_{l,m;x}; = m_x^{-1} \circ l_x \colon F_x \to F_x$. By means of the morphisms $\varphi_{l,m} = \varphi^{-1}_{m,l}$ the formalism can be transformed from a particular choice of $\{l_x\}$ to any other one $\{m_x\}$. Running some steps ahead, we have to say

[9] As the fibres $F_x \subset F$ are, generally, infinitely dimensional, the dimension of F is generically infinity. On the theory of such manifolds, see, for instance, [41].

[10] In the most applications M is supposed to be of class C^2 or C^3 (e.g. the Riemannian manifold of general relativity) or even C^∞ (e.g. the Minkowski space-time of special relativity or the Euclidean space of classical/quantum mechanics).

[11] the last condition is equivalent on (F, π, M) to be fixed a path-independent, of class C^1, and isometric linear transport along paths (*vide infra* Subsect. 3.2). From such position, the formalism will be studied elsewhere. Here we notice that the maps $l_{x\to y}$ (see (3.13)) define such a transport: if $\gamma \colon J \to M$ and $s, t \in J$, by proposition 3.1 the map $l \colon \gamma \mapsto l^\gamma$ with $l^\gamma \colon (s,t) \mapsto l_{\gamma(s)\to\gamma(t)} = l^{-1}_{\gamma(t)} \circ l_{\gamma(s)}$ is a linear transport in (F, π, M); it is obviously path-independent, of class C^1, and isometric as l_x are such.

that the set $\{l_x\}$ can not be fixed on the base of conventional quantum mechanics, its particular choice is external to it. In this sense $\{l_x\}$ is a free parameter in the bundle formulation of quantum mechanics. Regardless of this, as we shall see, the predictions of the resulting theory are independent of the concrete choice of $\{l_x\}$ and coincide with the ones of conventional quantum mechanics.

3.2. Linear transports along paths

The general theory of linear transports along paths in vector bundles is developed at length in [31,32]. In the present investigation we shall need only a few definitions and results from these papers when the bundle considered is a Hilbert one (*vide supra* definition 3.1). To their partial introduction and motivation is devoted the current section.

Let (E, π, B) be a complex[12] vector bundle (see Subsect. 3.1 or, e.g., [15, 44]) with bundle (total) space E, base B, projection $\pi\colon E \to B$, and isomorphic fibres $\pi^{-1}(x) \subset E$, $x \in B$. Let $\mathcal{E}$ be the (standard, typical) fibre of the bundle, i.e. a vector space to which all $\pi^{-1}(x)$, $x \in B$ are isomorphic. By J and $\gamma\colon J \to B$ we denote, respectively, a real interval and path in B.

Definition 3.2. A linear transport along paths in the bundle (E, π, B) is a map L assigning to any path $\gamma\colon J \to B$ a map L^γ, transport along γ, such that $L^\gamma\colon (s,t) \mapsto L^\gamma_{s\to t}$ where the map

$$L^\gamma_{s\to t}\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \qquad s,t \in J, \tag{3.18}$$

called transport along γ from s to t, has the properties:

$$L^\gamma_{s\to t} \circ L^\gamma_{r\to s} = L^\gamma_{r\to t}, \qquad r,s,t \in J, \tag{3.19}$$

$$L^\gamma_{s\to s} = \mathsf{id}_{\pi^{-1}(\gamma(s))}, \qquad s \in J, \tag{3.20}$$

$$L^\gamma_{s\to t}(\lambda u + \mu v) = \lambda L^\gamma_{s\to t} u + \mu L^\gamma_{s\to t} v, \qquad \lambda,\mu \in \mathbb{C}, \quad u,v \in \pi^{-1}(\gamma(s)), \tag{3.21}$$

where $\circ$ denotes composition of maps and id_X is the identity map of a set X.

Remark 3.1. Equations (3.19) and (3.20) mean that L is a *transport along paths* in the bundle (E, π, B) [37, definition 2.1], while (3.21) specifies that it is *linear* [37, equation (2.8)]. In the present paper only linear transports will be used.

This definition generalizes the concept of a parallel transport in the theory of (linear) connections (see [37,47] and the references therein for details and comparison).

[12] All of our definitions and results hold also for real vector bundles. Most of them are valid for vector bundles over more general fields too but this is inessential for the following.

A few comments on definition 3.2 are now in order. According to equation (3.18), a linear transport along paths may be considered as a path-depending connection: it establishes a fibre (isomorphic - see below) correspondence between the fibres over the path along which it acts. By virtue of equation (3.21) this correspondence is linear. Such a condition is a natural one when vector bundles are involved, it simply represents a compatibility condition with the vectorial structure of the bundle (see [37, sect. 2.3] for details). Equation (3.20) is a formal realization of our intuitive and naïve understanding that if we 'stand' at some point of a path without 'moving' along it, then 'nothing' should happen with the fibre over that point. This property fixes a 0-ary operation in the set of (linear) transports along paths, defining in it the 'unit' transport. At last, the equality (3.19), which may be called a group property of the (linear) transports along paths, is a rigorous expression of the intuitive representation that the 'composition' of two (linear) transports along one and the same path must be a (linear) transport along the same path.

In general, different forms of (3.18)–(3.21) are well know properties of the parallel transports generated by (linear) connections (see [47]). By this reason these transports turn to be special cases of the general (linear) transport along paths [47, theorem 3.1]. In particular, comparing definition 3.2 with [48, definition 2.1] and taking into account [48, proposition 4.1], we conclude that special types of linear transports along paths are: the parallel transport assigned to a linear connection (covariant differentiation) of the tensor algebra of a manifold [49, 50], Fermi-Walker transport [51, 52], Fermi transport [52], Truesdell transport [53, 54], Jaumann transport [55], Lie transport [50, 51], the modified Fermi-Walker and Frenet-Serret transports [56], etc. Consequently definition 3.2 is general enough to cover a list of important transports used in theoretical physics and mathematics. Thus studying the properties of the linear transports along paths, we can make corresponding conclusions for any one of the transports mentioned.[13]

From (3.19) and (3.20), we get that $L^{\gamma}_{s\to t}$ are invertible and

$$(L^{\gamma}_{s\to t})^{-1} = L^{\gamma}_{t\to s}, \qquad s,t \in J. \tag{3.22}$$

Hence the linear transports along paths are in fact linear isomorphisms between the fibres over the path along which they act.

The following two propositions establish the general structure of the linear transports along paths.

[13]The concept of linear transport along paths in vector bundles can be generalized to the transports along paths in arbitrary bundles [37] and to transports along maps in bundles [57]. An interesting considerations of the concept of (parallel) 'transport' (along closed paths) in connection with homotopy theory and the classification problem of bundles can be found in [58]. These generalizations will not be used in the present work.

Proposition 3.1. *A map* (3.18) *is a linear transport along* γ *from* s *to* t *for every* $s,t\in J$ *if and only if there exist an isomorphic with* $\pi^{-1}(x)$, $x\in B$ *vector space* V *and family of linear isomorphisms* $\{F(s;\gamma)\colon \pi^{-1}(\gamma(s))\to V,\ s\in J\}$ *such that*

$$L^{\gamma}_{s\to t}=F^{-1}(t;\gamma)\circ F(s;\gamma),\qquad s,t\in J. \tag{3.23}$$

Proof. If (3.18) is a linear transport along γ from s to t, then fixing some $s_0\in J$ and using (3.20) and (3.22), we get $L^{\gamma}_{s\to t}=L^{\gamma}_{s_0\to t}\circ L^{\gamma}_{s\to s_0}=\bigl(L^{\gamma}_{t\to s_0}\bigr)^{-1}\circ L^{\gamma}_{s\to s_0}$. So (3.23) holds for $V=\pi^{-1}(\gamma(s_0))$ and $F(s;\gamma)=L^{\gamma}_{s\to s_0}$. Conversely, if (3.23) is valid for some linear isomorphisms $F(s;\gamma)$, then a straightforward calculation shows that it converts (3.19) and (3.20) into identities and (3.21) holds due to the linearity of $F(s;\gamma)$. □

Proposition 3.2. *Let in the vector bundle* (E,π,B) *be given linear transport along paths with a representation* (3.23) *for some vector space* V *and linear isomorphisms* $F(s;\gamma)\colon \pi^{-1}(\gamma(s))\to V$, $s\in J$. *Then for a vector space* ${}^{\star}V$ *there exist linear isomorphisms* ${}^{\star}F(s;\gamma)\colon \pi^{-1}(\gamma(s))\to {}^{\star}V$, $s\in J$ *for which*

$$L^{\gamma}_{s\to t}={}^{\star}F^{-1}(t;\gamma)\circ {}^{\star}F(s;\gamma),\qquad s,t\in J. \tag{3.24}$$

iff there exists a linear isomorphism $D(\gamma)\colon V\to {}^{\star}V$ *such that*

$${}^{\star}F(s;\gamma)=D(\gamma)\circ F(s;\gamma),\qquad s\in J. \tag{3.25}$$

Proof. If (3.25) holds, then the substitution of $F(s;\gamma)=D^{-1}(\gamma)\circ {}^{\star}F(s;\gamma)$ into (3.23) results in (3.24). Vice versa, if (3.24) is valid, then from its comparison with (3.23) follows that $D(\gamma)={}^{\star}F(t;\gamma)\circ\bigl(F(t;\gamma)\bigr)^{-1}={}^{\star}F(s;\gamma)\circ\bigl(F(s;\gamma)\bigr)^{-1}$ is the required (independent of $s,t\in J$) isomorphism. □

Let (E,π,B) be a vector bundle whose bundle space E is a C^1 differentiable manifold. A linear transport L^{γ} along $\gamma\colon J\to B$ is called *differentiable of class* C^k, $k=0,1$, or simply C^k transport, if for arbitrary $s\in J$ and $u\in\pi^{-1}(\gamma(s))$, the path $\overline{\gamma}_{s;u}\colon J\to E$ with $\overline{\gamma}_{s;u}(t):=L^{\gamma}_{s\to t}u\in\pi^{-1}(\gamma(t))$, $t\in J$, is a C^k mapping in the bundle space E.[14] If a C^k linear transport has a representation (3.23), the mapping $s\mapsto F(s;\gamma)$ is of class C^k. So, the transport L^{γ} is of class C^k iff $L^{\gamma}_{s\to t}$ has C^k dependence on s and t simultaneously. If $\{e_i(\cdot;\gamma)|i=1,\dots,\dim\pi^{-1}(\gamma(s))\}$ is a C^k frame along γ, i.e. $\{e_i(s;\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$ and the mapping $s\mapsto e_i(s;\gamma)$ is of class C^k for every i, L^{γ} is of class C^k iff its matrix $\boldsymbol{L}(t,s;\gamma)$ with respect to $\{e_i(s;\gamma)\}$, $s\in J$ has C^k dependence on s and t. Here the elements of $\boldsymbol{L}(t,s;\gamma)$ are defined via the expansion

$$L^{\gamma}_{s\to t}\bigl(e_i(s;\gamma)\bigr)=:L^{j}{}_{i}(t,s;\gamma)e_j(t;\gamma)\qquad s,t\in J. \tag{3.26}$$

[14] If E is of class C^r with $r=0,1,\dots,\infty,\omega$, we can define in an evident way a C^k transport for every $k\le r$.

A transport L along paths in (E, π, B), E being C^1 manifold, is said to be of class C^k, $k = 0, 1$, if the corresponding transport L^γ along γ is of class C^k for every C^1 path $\gamma\colon J \to B$. Further we consider only C^1 linear transports along paths whose matrices will be referred to smooth frames along paths.

The above definition and results for linear transports along paths deal with the general case concerning arbitrary vector bundles and are therefore insensitive to the dimensionality of the bundle's base or fibres. Below we point out some peculiarities of the case of a Hilbert bundle whose fibres are generally infinitely dimensional.

For linear transports in a Hilbert bundle are valid all results of [31,32,37] with a possible exception of the ones in which (local) bases in the fibres are involved. The cause for this is that the dimension of a Hilbert space is (generally) infinity. So, there arise problems connected with the convergence or divergence of the corresponding sums or integrals. Below we try to avoid these problems and to formulate our assertions and results in an invariant way.

Of course, propositions 3.1 and 3.2 remain valid on Hilbert bundles; the only addition is that the vector spaces V and ${}^\star V$ are now Hilbert spaces.

In [31, sect. 3] are introduced the so-called *normal* frames for a linear transport along paths as a (local) field of bases in which (on some set) the matrix of the transport is unit. Further in this work, in subsection 8.2, we shall see that the normal frames realize the Heisenberg picture of motion in the Hilbert bundle formulation of quantum mechanics.

Now (see below the paragraph after equation (3.28)) we shall establish a result specific for the Hilbert bundles that has no analogue in the general theory: a transport along paths is Hermitian if and only if it is unitary. This assertion is implicitly contained in [1, sect. 3] (see the paragraph after equation (3.6) in it).

We call a (possibly linear) transport along paths in (F, π, M) *Hermitian* or *unitary* if it satisfies respectively (3.11) or (3.12) in which x, and y are replaced with arbitrary values of the parameter of the transportation path, i.e. if respectively

$$(L^\gamma_{s\to t})^\ddagger = L^\gamma_{s\to t}, \qquad s, t \in J, \quad \gamma\colon J \to M, \tag{3.27}$$

$$(L^\gamma_{s\to t})^\ddagger = (L^\gamma_{t\to s})^{-1}. \tag{3.28}$$

A simple corollary from (3.22) is the equivalence of (3.27) and (3.28); therefore, a *transport along paths in a Hilbert bundle is Hermitian if and only if it is unitary*, i.e. these concepts are equivalent. For such transports we say that they are *consistent* or *compatible* with the Hermitian structure (metric (inner product)) of the Hilbert bundle [59]. Evidently, they are *isometric* fibre maps along the paths they act. Therefore, a transport along

paths in a Hilbert bundle is isometric iff it is Hermitian or iff it is unitary.[15]

3.3. Liftings of paths, sections, and derivations along paths

A *lifting*[16] (in a vector bundle (E,π,B)) of a map $g\colon X\to B$, X being a set, is a map $\overline{g}\colon X\to E$ such that $\pi\circ\overline{g}=g$; in particular, the liftings of the identity map id_B of the base B are called *sections* and their set is $\mathrm{Sec}(E,\pi,B):=\{\sigma|\sigma\colon B\to E,\ \pi\circ\sigma=\mathsf{id}_B\}$. Let $\mathrm{P}(A):=\{\gamma|\gamma\colon J\to A\}$ be the *set of paths* in a set A and $\mathrm{PLift}(E,\pi,B):=\{\lambda|\lambda\colon \mathrm{P}(B)\to \mathrm{P}(E),\ (\pi\circ\lambda)(\gamma)=\gamma \text{ for } \gamma\in\mathrm{P}(B)\}$ be the *set of liftings of paths* from B to E.[17] The set $\mathrm{PLift}(E,\pi,B)$ is: (i) A natural $\mathbb{C}$-vector space if we put $(a\lambda+b\mu)\colon\gamma\mapsto a\lambda_\gamma+b\mu_\gamma$ for $a,b\in\mathbb{C}$, $\lambda,\mu\in\mathrm{PLift}(E,\pi,B)$, and $\gamma\in\mathrm{P}(B)$, where, for brevity, we write λ_γ for $\lambda(\gamma)$, $\lambda\colon\gamma\mapsto\lambda_\gamma$; (ii) A natural left module with respect to complex functions on B: if $f,g\colon B\to\mathbb{C}$, we define $(f\lambda+g\mu)\colon\gamma\mapsto(f\lambda)_\gamma+(g\mu)_\gamma$ with $(f\lambda)_\gamma(s):=f(\gamma(s))\lambda_\gamma(s)$ for $\gamma\colon J\to B$ and $s\in J$; (iii) A left module with respect to the set $\mathrm{PF}(B):=\{\varphi|\varphi\colon\gamma\mapsto\varphi_\gamma,\ \gamma\colon J\to B,\ \varphi_\gamma\colon J\to\mathbb{C}\}$ of *functions along paths* in the base B: for $\varphi,\psi\in\mathrm{PF}(B)$, we set $(\varphi\lambda+\psi\mu)\colon\gamma\mapsto(\varphi\lambda)_\gamma+(\psi\mu)_\gamma$ where $(\varphi\lambda)_\gamma(s):=(\varphi_\gamma\lambda_\gamma)(s):=\varphi_\gamma(s)\lambda_\gamma(s)$.

The dimension of $\mathrm{PLift}(E,\pi,B)$ as a $\mathbb{C}$-vector space is infinity but as a left $\mathrm{PF}(B)$-module is equal to the one of (E,π,B) (i.e. of its fibres). In the last case a basis in $\mathrm{PLift}(E,\pi,B)$ can be constructed as follows. For every $\gamma\colon J\to B$ and $s\in J$, choose a basis $\{e_i(s;\gamma)|i=1,\dots,\dim\pi^{-1}(\gamma(s))\}$ in $\pi^{-1}(\gamma(s))$; if E is a C^1 manifold, we suppose $e_i(s;\gamma)$ to have a C^1 dependence on s. Define $e_i\in\mathrm{PLift}(E,\pi,B)$ by $e_i\colon\gamma\mapsto e_i|_\gamma:=e_i(\cdot;\gamma)$, i.e. $e_i|_\gamma\colon s\mapsto e_i|_\gamma(s):=e_i(s;\gamma)$. The set $\{e_i\}$ is a basis in $\mathrm{PLift}(E,\pi,B)$, i.e. for every $\lambda\in\mathrm{PLift}(E,\pi,B)$ there are $\lambda^i\in\mathrm{PF}(B)$ such that $\lambda=\sum_i\lambda^ie_i$ and $\{e_i\}$ are $\mathrm{PF}(B)$-linearly independent. Actually, for $\gamma\colon J\to B$ and $s\in J$, we have $\lambda_\gamma(s)\in\pi^{-1}(\gamma(s))$, so there exists numbers $\lambda^i_\gamma(s)\in\mathbb{C}$ such that $\lambda_\gamma(s)=\sum_i\lambda^i_\gamma(s)e_i(s;\gamma)$. Defining $\lambda^i\in\mathrm{PF}(B)$ by $\lambda^i\colon\gamma\mapsto\lambda^i_\gamma$ with $\lambda^i_\gamma\colon s\mapsto\lambda^i_\gamma(s)$, we get $\lambda=\sum_i\lambda^ie_i$; if $e_i(\cdot;\gamma)$ is of class C^1, such are λ^i_γ. The $\mathrm{PF}(B)$-linear independence of $\{e_i\}$ is evident corollary of the $\mathbb{C}$-linear independence of $\{e_i(s;\gamma)\}$. As we notice above, if E is C^1 manifold, we choose e_i, i.e. $e_i|_\gamma$, to be C^1 and, consequently, the components λ^i, i.e. λ^i_γ, are of class C^1 too.

Let (E,π,B) be a vector bundle whose bundle space E is C^1 manifold. A lift $\lambda\in\mathrm{PLift}(E,\pi,B)$ is said to be of class C^k, $k=0,1$, if in some (and hence in any) C^k frame in $\mathrm{PLift}(E,\pi,B)$ its components are of class C^k along any C^k path, i.e. λ is of class C^k if λ_γ is of class C^k for every C^k path

[15]The author thanks prof. James Stasheff (Math-UNC, Chapel Hill, NC, USA) for suggesting in July 1998 the term 'isometric transport' in the context given.

[16]For detail see, e.g., [46].

[17]Every linear transport L along paths provides a lifting of paths: for every $\gamma\colon J\to B$ fix some $s\in J$ and $u\in\pi^{-1}(\gamma(s))$, the mapping $\gamma\mapsto\overline{\gamma}_{s;u}$ with $\overline{\gamma}_{s;u}(t):=L^\gamma_{s\to t}u$, $t\in J$ is a lifting of paths from B to E.

γ. Analogously, $\varphi \in \mathrm{PF}(B)$ is of class C^k if φ_γ is of class C^k for a C^k path γ. Denote by $\mathrm{PLift}^k(E,\pi,B)$, $k=0,1$, the set of C^k liftings of paths from B to E and by $\mathrm{PF}^k(B)$, $k=0,1$, the set of C^k functions along paths in B. If also the base B is C^1 manifold, we denote by $\mathrm{Sec}^k(E,\pi,B)$ the set of C^k sections of the bundle (E,π,B).

Definition 3.3. A derivation along paths in (E,π,B) or a derivation of liftings of paths in (E,π,B) is a map

$$D\colon\ \mathrm{PLift}^1(E,\pi,B) \to \mathrm{PLift}^0(E,\pi,B) \tag{3.29a}$$

which is $\mathbb{C}$-linear,

$$D(a\lambda+b\mu) = aD(\lambda)+bD(\mu) \tag{3.30a}$$

for $a,b\in\mathbb{C}$ and $\lambda,\mu\in\mathrm{PLift}^1(E,\pi,B)$, and the mapping

$$D^\gamma_s\colon\ \mathrm{PLift}^1(E,\pi,B) \to \pi^{-1}(\gamma(s)), \tag{3.29b}$$

defined via $D^\gamma_s(\lambda) := \big((D(\lambda))(\gamma)\big)(s) = (D\lambda)_\gamma(s)$ and called derivation along $\gamma\colon J\to B$ at $s\in J$, satisfies the 'Leibnitz rule':

$$D^\gamma_s(f\lambda) = \frac{\mathrm{d}f_\gamma(s)}{\mathrm{d}s}\lambda_\gamma(s) + f_\gamma(s)D^\gamma_s(\lambda) \tag{3.30b}$$

for every $f\in\mathrm{PF}^1(B)$. The mapping

$$D^\gamma\colon\ \mathrm{PLift}^1(E,\pi,B) \to \mathrm{P}\big(\pi^{-1}(\gamma(J))\big), \tag{3.29c}$$

defined by $D^\gamma(\lambda) := (D(\lambda))|_\gamma = (D\lambda)_\gamma$, is called derivation along γ.

Before continuing with the study of linear transports along paths, we want to say a few words on the links between sections (along paths) and liftings of paths.

The set $\mathrm{PSec}(E,\pi,B)$ of *sections along paths* of (E,π,B) consists of mappings $\boldsymbol{\sigma}\colon \gamma\mapsto\boldsymbol{\sigma}_\gamma$ assigning to every path $\gamma\colon J\to B$ a section $\boldsymbol{\sigma}_\gamma\in\mathrm{Sec}\big((E,\pi,B)|_{\gamma(J)}\big)$ of the restricted on $\gamma(J)$ bundle. Every (ordinary) section $\sigma\in\mathrm{Sec}(E,\pi,B)$ generates a section $\boldsymbol{\sigma}$ along paths via $\boldsymbol{\sigma}\colon\gamma\mapsto\boldsymbol{\sigma}_\gamma := \sigma|_{\gamma(J)}$, i.e. $\boldsymbol{\sigma}_\gamma$ is simply the restriction of σ on $\gamma(J)$; hence $\boldsymbol{\sigma}_\alpha=\boldsymbol{\sigma}_\gamma$ for every path $\alpha\colon J_\alpha\to B$ with $\alpha(J_\alpha)=\gamma(J)$. Every $\boldsymbol{\sigma}\in\mathrm{PSec}(E,\pi,B)$ generates a lifting $\hat{\boldsymbol{\sigma}}\in\mathrm{PLift}(E,\pi,B)$ by $\hat{\boldsymbol{\sigma}}\colon\gamma\mapsto\hat{\boldsymbol{\sigma}}_\gamma := \boldsymbol{\sigma}_\gamma\circ\gamma$; in particular, the lifting $\hat\sigma$ associated to $\sigma\in\mathrm{Sec}(E,\pi,B)$ is given via $\hat\sigma\colon\gamma\mapsto\hat\sigma_\gamma=\sigma|_{\gamma(J)}\circ\gamma$.

Every derivation D along paths generates a map

$$\overline{D}\colon\ \mathrm{PSec}^1(E,\pi,B)\to\mathrm{PLift}^0(E,\pi,B),$$

which may be called a *derivation of C^1 sections along paths*, such that if $\boldsymbol{\sigma}\in\mathrm{PSec}^1(E,\pi,B)$, then $\overline{D}\colon\boldsymbol{\sigma}\mapsto\overline{D}\boldsymbol{\sigma}=\overline{D}(\boldsymbol{\sigma})$ where $\overline{D}\boldsymbol{\sigma}\colon\gamma\mapsto\overline{D}^\gamma\boldsymbol{\sigma}$

is a lifting of paths paths defined by $\overline{D}^{\gamma}\boldsymbol{\sigma}\colon s \mapsto (\overline{D}^{\gamma}\boldsymbol{\sigma})(s) := D_s^{\gamma}\hat{\boldsymbol{\sigma}}$ with $\hat{\boldsymbol{\sigma}}$ being the lifting generated by $\boldsymbol{\sigma}$, i.e. $\gamma \mapsto \hat{\boldsymbol{\sigma}}_{\gamma} := \boldsymbol{\sigma}_{\gamma} \circ \gamma$. Notice, if $\gamma\colon J \to B$ has self-intersections points and $x_0 \in \gamma(J)$ is such a point, the map $\gamma(J) \to \pi^{-1}(\gamma(J))$ given by $x \mapsto \{D_s^{\gamma}(\hat{\boldsymbol{\sigma}})|\gamma(s) = x,\ s \in J\}$, $x \in \gamma(J)$, is generally multiple-valued at x_0 and, consequently it is not a section of $(E, \pi, B)|_{\gamma(J)}$.

If B is a C^1 manifold and for some $\gamma\colon J \to B$ exists a subinterval $J' \subseteq J$ on which the restricted path $\gamma|J\colon J' \to B$ is without self-intersections, i.e. $\gamma(s) \neq \gamma(t)$ for $s, t \in J'$ and $s \neq t$, we can define the *derivation along* γ *of sections* over $\gamma(J')$ as a map

$$\mathsf{D}^{\gamma}\colon \operatorname{Sec}^1\big((E, \pi, B)|_{\gamma(J')}\big) \to \operatorname{Sec}^0\big((E, \pi, B)|_{\gamma(J')}\big) \tag{3.31}$$

such that

$$(\mathsf{D}^{\gamma}\sigma)(x) := D_s^{\gamma}\hat{\sigma} \qquad \text{for } x = \gamma(s) \tag{3.32}$$

where $s \in J'$ is unique for a given x and $\hat{\sigma} \in \operatorname{PLift}\big((E, \pi, B)|_{\gamma(J')}\big)$ is given by $\hat{\sigma} = \sigma|_{\gamma(J')} \circ \gamma|_{J'}$. Generally the map (3.31) defined by (3.32) is multiple-valued at the points of self-intersections of γ, if any, as $(\mathsf{D}^{\gamma}\sigma)(x) := \{D_s^{\gamma}\hat{\sigma} : s \in J,\ \gamma(s) = x\}$. The so-defined map $\mathsf{D}\colon \gamma \mapsto \mathsf{D}^{\gamma}$ is called *section-derivation along paths*. As we said, it is single-valued only along paths without self-intersections.

Generally a section along paths or lifting of paths does not define a (single-value) section of the bundle as well as to a lifting along paths there does not correspond some (single-value) section along paths. The last case admits one important special exception: if a lifting λ is such that the lifted path λ_{γ} is an 'exact topological copy' of the underlying path $\gamma\colon J \to B$, i.e. if there exist $s, t \in J$, $s \neq t$ for which $\gamma(s) = \gamma(t)$, then $\lambda_{\gamma}(s) = \lambda_{\gamma}(t)$. Such a lifting λ generates a section $\overline{\lambda} \in \operatorname{PSec}(E, \pi, B)$ along paths given by $\overline{\lambda}\colon \gamma \mapsto \overline{\lambda}_{\gamma}$ with $\overline{\lambda}\colon \gamma(s) \mapsto \lambda_{\gamma}(s)$. In the general case, the mapping $\gamma(s) \mapsto \lambda_{\gamma}(s)$ for a lifting λ of paths is multiple-valued at the points of self-intersection of $\gamma\colon J \to B$, if any; for injective path γ this map is a section of $(E, \pi, B)|_{\gamma(J)}$. Such mappings will be called *multiple-valued sections along paths*.

With every derivation D along paths in (E, π, B) can be associated a derivation $\widetilde{D}$ along paths in $\operatorname{mor}_B(E, \pi, B)$. For this end every lifting $\operatorname{PLift}(\operatorname{mor}(E, \pi, B))$ should be regarded as a map

$$A\colon \operatorname{PLift}(E, \pi, B) \to \operatorname{PLift}(E, \pi, B) \tag{3.33}$$

such that, if $\lambda \in \operatorname{PLift}(E, \pi, B)$, $\gamma\colon J \to B$ and $s \in J$, then

$$A\colon \lambda \mapsto A(\lambda)\colon \gamma \mapsto (A(\lambda))_{\gamma} = A_{\gamma}(\lambda_{\gamma}), \quad A_{\gamma}(\lambda_{\gamma})\colon s \mapsto A_{\gamma}(s)(\lambda_{\gamma}(s)). \tag{3.34}$$

For every derivation D along paths in (E,π,B), we define

$$\widetilde{D}\colon \mathrm{PLift}^1(\mathrm{mor}_B(E,\pi,B)) \to \mathrm{PLift}^0(\mathrm{mor}_B(E,\pi,B)) \tag{3.35}$$

by

$$\widetilde{D}\colon A \mapsto \widetilde{D}(A) := D\circ A \tag{3.36}$$

where $A\in \mathrm{PLift}^1(E,\pi,B)$ is considered as a map (3.33). Putting

$$\widetilde{D}^\gamma(A) := D^\gamma\circ A, \quad \widetilde{D}^\gamma_s(A) := D^\gamma_s\circ A, \tag{3.37}$$

it is a trivial verification that the map $\widetilde{D}$ is a derivation along paths in $\mathrm{mor}_B(E,\pi,B)$. The map $\widetilde{D}$ will be called *induced (from D) derivation along paths*.

Definition 3.4. The derivation D along paths generated by a C^1 linear transport L along paths is a map of type (3.29a) assigning to every path $\gamma\colon J\to B$ a map D^γ, derivation along γ generated by L, such that $D^\gamma\colon s\mapsto D^\gamma_s$, $s\in J$, is a map (3.29b), called derivation along γ at s assigned to L, given via

$$D^\gamma_s(\lambda) := \lim_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon}\big[L^\gamma_{s+\varepsilon\to s}\lambda_\gamma(s+\varepsilon)-\lambda_\gamma(s)\big]\Big\} \tag{3.38}$$

for every lifting $\lambda\in\mathrm{PLift}^1(E,\pi,B)$ with $\lambda\colon\gamma\mapsto\lambda_\gamma$.

Remark 3.2. The operator D^γ_s is an analogue of the covariant differentiation assigned to a linear connection; cf., e.g., [60, p. 139, equation (12)].

Remark 3.3. Notice, if γ has self-intersections and $x_0\in\gamma(J)$ is such a point, the mapping $x\mapsto\pi^{-1}(x)$, $x\in\gamma(J)$ given by $x\mapsto\{D^\gamma_s(\lambda)|\gamma(s)=x,\ s\in J\}$ is, generally, multiple-valued at x_0.

Let L be a linear transport along paths in (E,π,B). For every path $\gamma\colon J\to B$ choose some $s_0\in J$ and $u_0\in\pi^{-1}(\gamma(s_0))$. The mapping

$$\overline{L}\colon\gamma\mapsto\overline{L}^\gamma_{s_0,u_0},\quad \overline{L}^\gamma_{s_0,u_0}\colon J\to E,\quad \overline{L}^\gamma_{s_0,u_0}\colon t\mapsto\overline{L}^\gamma_{s_0,u_0}(t) := L^\gamma_{s_0\to t}u_0 \tag{3.39}$$

is, evidently, a lifting of paths.

Definition 3.5. The lifting of paths $\overline{L}$ from B to E in (E,π,B) defined via (3.39) is called lifting (of paths) generated by the (linear) transport L.

Equations (3.20) and (3.23), combined with (3.38), immediately imply

$$D^\gamma_t(\overline{L})\equiv 0,\qquad t\in J, \tag{3.40}$$

$$D^\gamma_s(a\lambda+b\mu) = aD^\gamma_s\lambda + bD^\gamma_s\mu,\qquad a,b\in\mathbb{C},\quad \lambda,\mu\in\mathrm{PLift}^1(E,\pi,B), \tag{3.41}$$

where $s_0 \in J$ and $u(s) = L^{\gamma}_{s_0 \to s} u_0$ are fixed. In other words, equation (3.40) means that the lifging $\overline{L}$ is constant along every path γ with respect to D.

Let $\{e_i(s;\gamma)\}$ be a field of smooth bases along $\gamma\colon J \to B,\ s \in J$. Combining the linearity of L with (3.26) and (3.38), we find the explicit local action of D^{γ}_s:[18]

$$D^{\gamma}_s \lambda = \sum_i \left[\frac{\mathrm{d}\lambda^i_{\gamma}(s)}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\lambda^j_{\gamma}(s) \right] e_i(s;\gamma). \tag{3.42}$$

Here the *(2-index) coefficients* $\Gamma^i{}_j$ of the linear transport L are defined by

$$\Gamma^i{}_j(s;\gamma) := \left. \frac{L^i{}_j(s,t;\gamma)}{\partial t} \right|_{t=s} = -\left. \frac{L^i{}_j(s,t;\gamma)}{\partial s} \right|_{t=s} \tag{3.43}$$

and, evidently, uniquely determine the generated by L derivation D along paths.

A trivial corollary of (3.41) and (3.42) is the assertion that the derivation along paths generated by a linear transport is actually a derivation along paths (see definition 3.3).

If the transport's matrix $\boldsymbol{L}$ has a representation

$$\boldsymbol{L}(t,s;\gamma) = \boldsymbol{F}^{-1}(t;\gamma)\boldsymbol{F}(s;\gamma) \tag{3.44}$$

for some non-degenerate matrix-valued function $\boldsymbol{F}$, which is a corollary of (3.23), from (3.43), we get

$$\boldsymbol{\Gamma}(s;\gamma) := \left[\Gamma^i{}_j(s;\gamma)\right] = \left. \frac{\partial \boldsymbol{L}(s,t;\gamma)}{\partial t} \right|_{t=s} = \boldsymbol{F}^{-1}(s;\gamma)\frac{\mathrm{d}\boldsymbol{F}(s;\gamma)}{\mathrm{d}s}. \tag{3.45}$$

From here and (3.43), we see that the change $\{e_i\} \to \{e'_i = \sum_j A^j_i e_j\}$ of the local bases along γ with a nondegenerate C^1 matrix $A := \left[A^j_i\right]$ implies

$$\boldsymbol{\Gamma}(s;\gamma) = \left[\Gamma^i{}_j(s;\gamma)\right] \mapsto \boldsymbol{\Gamma}'(s;\gamma) = \left[\Gamma'^i{}_j(s;\gamma)\right]$$

with

$$\boldsymbol{\Gamma}'(s;\gamma) = A^{-1}(s;\gamma)\boldsymbol{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s}. \tag{3.46}$$

It is a fundamental result [31,32] that there exists a bijective correspondence between linear transports along paths and derivations along paths: a linear transport generates derivation via (3.38) and, *vice versa*, for every

[18] Here and below we suppose the existence of derivatives like $\mathrm{d}\lambda^i_{\gamma}(s)/\mathrm{d}s$, viz. $\lambda^i_{\gamma}\colon J \to \mathbb{C}$ to be a C^1 mapping. This, of course, imposes some smoothness conditions on γ which we assume to hold. Evidently, for the purpose γ must be at least continuous. Without going into details, we notice that the most natural requirement for γ, when B is a manifold, is to admit it to be a C^1 map.

derivation along paths exists a unique transport generating it by (3.38). Locally this correspondence is established by the coincidence of the transport's and derivation's coefficients.[19]

Every transport L along paths in a vector bundle (E,π,B) generates a linear transport ${}^{\circ}L$ along paths in the bundle $\mathrm{mor}_B(E,\pi,B)$ of point-restricted morphisms over B in (E,π,B). If $\gamma\colon J\to B$, explicitly we have [59, equations (3.9)–(3.12)] ${}^{\circ}L\colon \gamma\mapsto {}^{\circ}L^{\gamma}\colon (s,t)\mapsto {}^{\circ}L^{\gamma}_{s\to t}$ $s,t\in J$ with

$$\begin{aligned}{}^{\circ}L^{\gamma}_{s\to t}(\varphi_{\gamma(s)}) &:= L^{\gamma}_{s\to t}\circ\varphi_{\gamma(s)}\circ L^{\gamma}_{t\to s}\\ &\in (\pi_0^B)^{-1}(\gamma(t)) = \{\psi|\psi\colon \pi^{-1}(\gamma(t))\to\pi^{-1}(\gamma(t))\}\end{aligned}\tag{3.47}$$

for every $\varphi_{\gamma(s)}\colon \pi^{-1}(\gamma(s))\to\pi^{-1}(\gamma(s))$. The transport ${}^{\circ}L$ will be called *associated* to L (in $\mathrm{mor}_B(E,\pi,B)$).

The generated by ${}^{\circ}L$ derivation along paths in $\mathrm{mor}_B(E,\pi,B)$ will be denoted by ${}^{\circ}D$ and called *derivation associated to the derivation* D generated by L. So, if $A\in \mathrm{PLift}^1(\mathrm{mor}_B(E,\pi,B))$, then

$${}^{\circ}D^{\gamma}_s(A) := \lim_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon}\big[\,{}^{\circ}L^{\gamma}_{s+\varepsilon\to s}A_{\gamma}(s+\varepsilon) - A_{\gamma}(s)\big]\Big\}.\tag{3.48}$$

If the lift A_γ of $\gamma\colon J\to B$ in the bundle space E_0^B of $\mathrm{mor}_B(E,\pi,B)$ is *linear* and the matrix of A_γ in $\{e_i(s;\gamma)\}$ is $\boldsymbol{A}_\gamma(s)$, then from (3.42)–(3.45), one finds the explicit matrix of ${}^{\circ}D^{\gamma}_s A$ as

$$\big[\,{}^{\circ}D^{\gamma}_s A\big] = [\boldsymbol{\Gamma}(s;\gamma),\boldsymbol{A}_\gamma(s)]_{-} + \frac{\mathrm{d}\boldsymbol{A}_\gamma(s)}{\mathrm{d}s}\tag{3.49}$$

where $[\cdot,\cdot]_{-}$ means the commutator of matrices and the equality $\boldsymbol{L}(s,s;\gamma)=\mathbb{1}$, $\mathbb{1}$ being the unit matrix, was used (see (3.44) or (3.20)).

Under some assumptions, the matrix of the induced derivative $(\widetilde{D}^{\gamma}_s A)\lambda = D^{\gamma}_s(A(\lambda))$ is (see (3.37) and (3.42))

$$\big[(\widetilde{D}^{\gamma}_s A)\lambda\big] = \frac{\mathrm{d}\boldsymbol{A}_\gamma(s)}{\mathrm{d}s}\boldsymbol{\lambda}_\gamma(s) + \boldsymbol{A}_\gamma(s)\frac{\mathrm{d}\boldsymbol{\lambda}_\gamma(s)}{\mathrm{d}s} + \boldsymbol{\Gamma}(s;\gamma)\boldsymbol{A}_\gamma(s)\boldsymbol{\lambda}_\gamma(s)\tag{3.50}$$

where $\lambda\in\mathrm{PLift}^1(E,\pi,B)$ is a C^1 lifting along paths in (E,π,B) and $\boldsymbol{\lambda}_\gamma(s)$ is the matrix of $\lambda_\gamma(s)$ in $\{e_i(s;\gamma)\}$.

In our investigation the above-presented general definitions and results will be applied to the particular case of a Hilbert bundle. Since its dimension is generically infinite, some problems connected with convergence of sums (which generally are integrals) or decompositions like $\sum_i \lambda^i e_i$ could arise. We shall comment on these problems in Sect. 6 of the present work.

[19] The coefficients (components) of derivation D along paths are defined by $D^{\gamma}_s e_i = \sum_j \Gamma^j{}_i(s;\gamma)e_j(s;\gamma)$.

4. The Hilbert bundle description of quantum mechanics

As we shall see in this investigation, the Hilbert bundles provide a natural mathematical framework for a geometrical formulation of quantum mechanics. In it all quantum-mechanical quantities, such as Hamiltonians, observables, wavefunctions, etc., have an adequate description. For instance, the evolution of a systems is described as an appropriate (parallel or, more precisely, linear) transport of system's state liftings of path or sections along paths. We have to emphasize on the fact that the new bundle formulation of quantum mechanics and the conventional one are completely equivalent at the present stage of the theory.

4.1. Brief literature overview

Several attempts have been made for a (partial) (re)formulation of non-relativistic quantum mechanics in terms of bundles. Works containing such material were mentioned in Sect. 1. Below are marked only those of them which directly or indirectly lead to some essential elements of our approach to quantum mechanics.

It seems for the first time the appropriate bundle approach to quantum mechanics was developed in [35] where the single Hilbert space of quantum mechanics is replaced with an infinitely many copies of it forming a bundle space over the 1-dimensional 'time' manifold (i.e. over $\mathbb{R}_+$). In this Hilbert fibre bundle the quantum evolution is (equivalently) described as a kind of 'parallel' transport of appropriate objects over the bundle's base.

Analogous construction, a Hilbert bundle over the system's phase space, is used in the Prugovečki's approach to quantum theory (see, e.g. the references in [29]).

The gauge, i.e. linear connection, structure in quantum mechanics is first mentioned [25]. That structure is pointed to be connected with the system's Hamiltonian. This observation will find natural explanation in our work (see [3]).

Some ideas concerning the interpretation of quantum evolution as a kind of a 'parallel' transport in a Hilbert bundle can also be found in [33, 36].

4.2. Motivation

Below are present some non-exactly rigorous ideas and statements whose only purpose is the *motivation* for applying the fibre bundle formalism to quantum mechanics. Another excellent arguments and motives confirming this approach are given in [35].

Let M be a differentiable manifold, representing in our context the space in which the (nonrelativistic) quantum-mechanical objects 'live', i.e. the

usual 3-dimensional coordinate space (isomorphic to $\mathbb{R}^3$ with the corresponding structures).[20] Let $\gamma\colon J \to M$, J being an $\mathbb{R}$-interval, be the trajectory of an observer describing the behaviour of a quantum system at any moment $t \in J$ by a state vector $\Psi_\gamma(t)$ depending on t and, possibly, on γ.[21] For a fixed point $x = \gamma(t) \in M$ the variety of state vectors describing a quantum system and corresponding to different observers form a Hilbert space $F_{\gamma(t)}$ which *depends on $\gamma(t) = x$, but not on γ and t separately.*[22]

Remark 4.1. As we said above in footnote 20, the next considerations are completely valid mathematically if M is an arbitrary differentiable manifold and γ is a path in it. In this sense M and γ are free parameters in our theory and their concrete choice is subjected only to *physical reasons*, first of all, ones requiring adequate physical interpretation of the resulting theory. (The arbitrariness of M in a similar construction is mentioned in [36, sect. I] too.) Typical candidates for M are: the 3-dimensional Euclidean space $\mathbb{E}^3$, $\mathbb{R}^3$, the 4-dimensional Minkowski space M^4 of special relativity or the Riemannian space V_4 of general relativity, the system's configuration or phase space, the 'time' manifold $\mathbb{R}_+ := \{a : a \in \mathbb{R},\ a > 0\}$, etc. Correspondingly, γ obtains interpretation as particle's trajectory, its world line, and so on. The degenerate case when M consists of a single point corresponds (up to an isomorphism - see below) to the conventional quantum mechanics. Throughout this work, we most often take $M = \mathbb{R}^3$ as a natural choice corresponding to the non-relativistic case investigated here but, as we said, this is not required by necessity. Elsewhere we shall see that $M = M^4$ or $M = V_4$ are natural choices in the relativistic region. An expanded commend on these problems will be given in Sect. 12 of this work. Here we want only to note that the interpretation of γ as an observer's (particle's) trajectory or world line, as accepted in this work, is reasonable but not necessary one. Maybe more adequate is to interpret γ as a mean (in quantum-mechanical sense) trajectory of some point particle but this does not change anything in the mathematical structure of the bundle approach proposed here.

The spaces $F_{\gamma(t)}$ must be isomorphic as, from physical view-point, they simply represent the possible variety of state vectors from different positions. In this way over M arises a natural bundle structure, viz. a *Hilbert bundle* (F, π, M) with a total space F, projection $\pi\colon F \to M$ and isomorphic fibres

[20] In the following M can naturally be considered also as the Minkowski space-time of special relativity. In this case the below-defined observer's trajectory γ is his world line. But we avoid this interpretation because only the nonrelativistic case is investigated here. It is important to be noted that mathematically all of what follows is valid in the case when by M is understood an arbitrary differentiable manifold. The physical interpretation of these cases will be given elsewhere.

[21] In this way we introduce the (possible) explicit dependence of the description of system's state on the concrete observer with respect to which it is determined.

[22] If there exists a global time, as in the nonrelativistic quantum mechanics, the parameter $t \in J$ can be taken as such. Otherwise by t we have to understand the local ('proper' or 'eigen-') time of a concrete observer.

$\pi^{-1}(x) := F_x$. Since F_x, $x \in M$ are isomorphic, there exists a Hilbert space $\mathcal{F}$ and (linear) isomorphisms $l_x \colon F_x \to \mathcal{F},\ x \in M$. Mathematically $\mathcal{F}$ is the typical (standard) fibre of (F, π, M). The maps $\Psi_\gamma \colon J \to \pi^{-1}(\gamma(J))$ can be considered as sections over any part of γ without self-intersections (see below).

Now a natural question arises: how the quantum evolution in time in the bundle constructed is described? There are two almost 'evident' ways to do this. On one hand, we can postulate the conventional quantum mechanics in every fibre F_x, i.e. the Schrödinger equation for the state vector $\Psi_\gamma(t) \in F_{\gamma(t)}$ with $F_{\gamma(t)}$ being (an isomorphic copy of) the system's Hilbert space. But the only thing one gets in this way is an isomorphic image of the usual quantum mechanics in any fibre over M. Therefore one can not expect some new results or descriptions in this direction (see below (4.2) and the comments after it). On the other hand, we can demand the ordinary quantum mechanics to be valid in the fibre $\mathcal{F}$ of the bundle (F, π, M). This means to identify $\mathcal{F}$ with the system's Hilbert space of states and to describe the quantum time evolution of the system via the vector

$$\psi(t) = l_{\gamma(t)}(\Psi_\gamma(t)) \in \mathcal{F} \tag{4.1}$$

which evolves according to (2.1) or (2.6). This approach is accepted in the present investigation. What we intend to do further, is, by using the basic relation (4.1), to 'transfer' the quantum mechanics from $\mathcal{F}$ to (F, π, M) or, in other words, to investigate the quantum evolution in terms of the vector $\Psi_\gamma(t)$ connected with $\psi(t)$ via (4.1). Since $l_x,\ x \in M$ are isomorphisms, both descriptions are *completely equivalent*. This equivalence resolves a psychological problem that may arise *prima facie*: the single Hilbert space $\mathcal{F}$ of standard quantum theory [7–11] is replaced with a, generally, infinite number copies F_x, $x \in M$ thereof (cf. [35]). In the present investigation we shall show that the merit one gains from this is an entirely geometrical reformulation of quantum mechanics in terms of Hilbert fibre bundles.

The evolution of a quantum system will be described in a fibre bundle (F, π, M) with fixed isomorphisms $\{l_x,\ x \in M\}$ such that $l_x \colon F_x \to \mathcal{F}$, where $\mathcal{F}$ is the Hilbert space in which the system's evolution is described through the usual Schrödinger picture of motion.

So, in the Schrödinger picture a quantum system is described by a state vector ψ in $\mathcal{F}$. Generally [38] ψ depends (maybe implicitly) on the observer with respect to which the evolution is studied[23] and it satisfies the Schrödinger equation (2.6). We shall refer to this representation of quantum mechanics as a *Hilbert space description*. In the new *(Hilbert fibre) bundle description*, which will be studied below, the linear isomorphisms

[23] Usually this dependence is not written explicitly, but it is always presented as actually t is the time with respect to a given observer.

$l_x\colon F_x = \pi^{-1}(x) \to \mathcal{F},\ x \in M$ are supposed arbitrarily fixed[24] and the quantum systems are described by a *state liftings of paths or sections along paths* Ψ of a bundle (F, π, M) whose typical fibre is the Hilbert space $\mathcal{F}$ (the same Hilbert space as in the Hilbert space description).

Generally, to any vector $\varphi \in \mathcal{F}$ there corresponds a unique (global) section $\overline{\Phi} \in \mathrm{Sec}(F, \pi, M)$ defined via

$$\overline{\Phi}\colon x \mapsto \overline{\Phi}_x := l_x^{-1}(\varphi) \in F_x, \qquad x \in M, \quad \varphi \in \mathcal{F}. \tag{4.2}$$

Consequently to a state vector $\psi(t) \in \mathcal{F}$ one can assign the (global) section $\overline{\Psi}(t)$, $\overline{\Psi}(t)\colon x \mapsto \overline{\Psi}_x(t) = l_x^{-1}(\psi(t)) \in F_x$ and thus obtaining in F_x for every $x \in M$ an isomorphic picture of (the evolution in) $\mathcal{F}$. But in this way one can not expect significantly new results as the evolution in $\mathcal{F}$ is simply replaced with the (linearly isomorphic to it) evolution in F_x for every arbitrary fixed $x \in M$.[25] This reflects the fact that the quantum mechanical description is defined up to linear isomorphism(s) (see note 4.4 below). Besides, on the contrary to the bundle description, in this way one looses the explicit dependence on the observer. So, in it one cannot get really new results with respect to the Hilbert space description.

4.3. Basic ideas and statement of the problems

Taking into account the (more or less heuristic) arguments from the previous subsection, we pose the following problem: given a Hilbert bundle (F, π, M) with the properties described in Subsect. 3.1 and a path $\gamma\colon J \to M$, describe the quantum evolution of some quantum system in this bundle provided the standard fibre $\mathcal{F}$ is the system's Hilbert space of states. For the moment, we identify the bundle's base M with the space-time model used: for definiteness we take for it the 3-dimensional Euclidean space $\mathbb{E}^3$. The path mentioned will be interpreted as a trajectory of a certain observer, correspondingly its parameter will be treated as a (global) time.

At precisely this point some natural questions arise. First of all, why should one replace the single conventional Hilbert space of the system with a Hilbert bundle, i.e. with (generally) infinite number of different 'local' Hilbert spaces each of which is associated to a single space point? And, why the introduced reference path is required? These are basic moments which *a posteriori* will be justified by the results but *a priori* their essence is in the following. Conventionally, the system's evolution is described by *different state vectors, one for every instant of time, in the unique Hilbert space of the system.* These state vectors *generically depends on the observer* with respect to which the system is explored. This dependence is often implicit one in

[24] The particular choice of $\{l_x\}$ (and, consequently, of the fibres F_x) is inessential for our investigation.

[25] The machinery of global sections like (4.2) is used in [36] for the bundle approach to quantum mechanics contained in this paper.

quantum mechanics but it is always presented: the states might be different because, e.g., the observers could have different velocities, or be rotated relative to each other. So, different observers assign, generally, different state vectors to one and the same quantum system at a given moment and *these vectors belong to the (initial) Hilbert space of the system.* In this context the shift to a Hilbert bundle pursuits a twofold goal: the *explicit observer-dependence* of the 'state vectors'[26] and the *split of the time values of the 'state vectors' into different Hilbert spaces.* We achieve this by describing the system's state at a time $t \in J$ with respect to observer with trajectory $\gamma\colon J \to M$ with a 'state vector' from the 'local' Hilbert space attached to the point $\gamma(t)$, i.e. *from the fibre* $F_{\gamma(s)} := \pi^{-1}(\gamma(s))$. Besides through γ, the observer-dependence of the 'state vectors' is introduced, maybe implicitly, via the Hamiltonian which does not exists *per ce* but is always given with respect to some concrete observer. Consequently, if we have two observers with trajectories $\alpha\colon J_\alpha \to M$ and $\beta\colon J_\beta \to M$ with $J_\alpha \cap J_\beta \neq \varnothing$, at a moment $t \in J_\alpha \cap J_\beta$ they will describe the state of a system via some vectors $\Psi_\alpha \in F_{\alpha(t)}$ and $\Psi_\beta \in F_{\beta(t)}$. In particular, if it happens that at the moment t the observers are at one and the same point $x = \alpha(t) = \beta(t)$, the 'state vectors' Ψ_α and Ψ_β will be from a single fibre, the one over x, i.e. the Hilbert space $F_x = \pi^{-1}(x)$. But generally these vectors will be different unless the observers are absolutely identical at the moment t.[27]

At the moment it is not clear what one gains from 'unwrapping' the time evolution from the single Hilbert space $\mathcal{F}$ to a collection $\{F_{\gamma(t)} | t \in J\}$ of 'local' Hilbert spaces along the observer's trajectory γ. We shall try to explain this in Subsect 4.4. In advance, we want only to state the main merit of the proposed approach: a self-consistent purely geometrical formulation of (nor-)relativistic quantum mechanics in terms of Hilbert bundles.

Now it is time for explicit rigorous statement of the basic assertions and the problems we are going to solve later in this investigation. Notice, if the opposite is not explicitly stated, we consider only pure quantum states that conventionally are described by vectors in a Hilbert space.

Postulate 4.1. *Let there be given a quantum system and $\mathcal{F}$ be its Hilbert space of states. To this system we assign a C^1 compatible Hilbert bundle (F, π, M) with bundle space F, projection $\pi\colon F \to M$, and base M. Besides, we suppose:*

(i) *The base M and the bundle space F are C^1 differentiable manifolds.*

(ii) *The point-trivializing (isometric) isomorphisms $l_x\colon \pi^{-1}(x) \to \mathcal{F}$, $x \in M$, are fixed and of class C^1. Their dependence on x is also required to be of class C^1, i.e. (F, π, M) is of class C^1.*

[26] Here we use inverted commas as, actually, the right term is bundle state vector, i.e. a state lifting or section at some point; *vide infra* in this section.

[27] For example, if the observers have non-zero relative acceleration at x, it is quite natural that they will assign different 'state vectors' to the system at the moment t.

(iii) *The (standard) fibre of (F, π, M) is the system's Hilbert space of states $\mathcal{F}$ in which the conventional quantum mechanics is valid.*

Note 4.1. It should be emphasize, here we introduce two parameters which are left free from the quantum mechanics and are external to it: the base M and the set of isomorphisms $\{l_x\}$. For the sake of physical interpretation (see remark 4.1), we identify M with the space(-time) model, in particular with the 3-dimensional Euclidean space $\mathbb{E}^3$ (or the Minkowski 4-dimensional space-time M^4, or the Riemannian 4-manifold V_4 of general relativity, etc.). This does not influence the basic scheme which is valid for arbitrary manifold M. What concerns the set $\{l_x\}$, in the present work we consider it as given and its analysis and interpretation will be given elsewhere. In this connection, we want to notice three things: (i) The arbitrariness in $\{l_x\}$ reflects the natural one in the choice of the system's Hilbert space of states which is defined up to isomorphism; (ii) As we shall see, the mathematical formalism depends on the choice of $\{l_x\}$ but the physically predictable results (the mean values (mathematical expectations) of the operators) do not; (iii) In another investigation we intend to show that on the base of the set $\{l_x\}$ of isomorphisms is very likely to be achieved a kind of unification of quantum mechanics and gravity.

Definition 4.1. The bundle (F, π, M) introduce via postulate 4.1 will be called Hilbert bundle (of states) of the quantum system, or simply system's Hilbert bundle (of states).

Postulate 4.2. *Let $J \subseteq \mathbb{R}$ be real interval representing the period of time in which a quantum system is investigated, (F, π, M) be its Hilbert bundle, and $\gamma\colon J \to M$ be a C^1 path in the base M. In (F, π, M) the state of the system at a moment $t \in J$ is described by a map Ψ assigning to a pair (γ, t) a vector $\Psi_\gamma(t) \in \pi^{-1}(\gamma(t)) = F_{\gamma(t)}$ such that*

$$\Psi_\gamma(t) = l^{-1}_{\gamma(t)}(\psi(t)) \in F_{\gamma(t)} \tag{4.3}$$

where $\psi(t) \in \mathcal{F}$ is the conventional state vector in the system's Hilbert space of states ($\equiv$ the bundle's fibre) describing the system's state at the moment t in the (usual) quantum mechanics.

Definition 4.2. The description of a quantum system via the map Ψ (resp. ψ) in the Hilbert bundle (F, π, M) (resp. Hilbert space $\mathcal{F}$) will be called Hilbert bundle (resp. Hilbert space) description (of the quantum mechanics of the system).

Note 4.2. Since the maps $l_x\colon F \to \mathcal{F}$ are isomorphisms, the description of the quantum states by Ψ and ψ is completely equivalent.

Note 4.3. As we said above, the path γ will be physically interpreted as a trajectory (or, possibly, world line) of an observer moving in M and with

respect to which the quantum system is studied (or 'who' investigates it). So, in the bundle description the 'state vector' $\Psi_\gamma(t)$, representing the system state at a moment t, explicitly depends on the observer which is depicted in the index γ in $\Psi_\gamma(t)$. This is contrary to the conventional quantum mechanics where this dependence is implicitly assumed almost everywhere. Thus we come to the above-mentioned situation: different observers describe the system's state at a fixed moment by vectors from, generally, different fibres of the bundle; these vectors belong to one and the same fibre over some point in M iff the observers happen to be simultaneously in it but, even in this case, the vectors need not to coincide, they are generically different unless the observers are absolutely identical.

Note 4.4. The bundle, as well as the conventional, description of quantum mechanics is defined up to a linear isomorphism(s). In fact, if $\imath\colon \mathcal{F} \to \mathcal{F}'$, $\mathcal{F}'$ being a Hilbert space, is a linear isomorphism (which may depend on the time t), then $\psi'(t) = \imath(\psi(t))$ equivalently describes the evolution of the quantum system in $\mathcal{F}'$. (Note that in this way, for $\mathcal{F}' = \mathcal{F}$, one can obtain the known pictures of motion in quantum mechanics — see [9]or Sect. 8.) In the bundle case the shift from $\mathcal{F}$ to $\mathcal{F}'$ is described by the transformation $l_x \to l'_x := \imath \circ l_x$ which reflects the arbitrariness in the choice of the typical fibre (now $\mathcal{F}'$ instead of $\mathcal{F}$) of (F, π, M). There is also arbitrariness in the choice of the fibres $F_x = \pi^{-1}(x)$ which is of the same character as the one in the case of $\mathcal{F}$, viz. if $\imath_x\colon F_x \to F'_x,\ x \in M$ are linear isomorphisms, then the fibre bundle (F', π', M) with $F' := \bigcup_{x\in M} F'_x,\ \pi'|_{F'_x} := \pi \circ \imath_x^{-1}$, typical fibre $\mathcal{F}$, and isomorphisms $l'_x := l_x \circ \imath_x^{-1}$ can equivalently be used to describe the state of a quantum system. In the most general case, we have a fibre bundle (F', π', M) with fibres $F'_x = \imath_x^{-1}(F_x)$, typical fibre $\mathcal{F}' = \imath(\mathcal{F})$, and isomorphisms $l'_x := \imath \circ l_x \circ \imath_x^{-1}\colon F'_x \to \mathcal{F}'$. Further we will not be interested in such generalizations. Thus, we shall suppose that all of the mentioned isomorphisms are fixed.

Let us now look on the mathematical nature of the map Ψ introduce via postulate 4.2. From one hand, as the notation suggests, the mapping $\Psi\colon \gamma \mapsto \Psi_\gamma$ with $\Psi_\gamma\colon t \mapsto \Psi_\gamma(t)$ is a lifting of paths, $\Psi \in \mathrm{PLift}(F, \pi, M)$, which is a trivial corollary of (4.3). On the other hand, we can consider Ψ as a multiple-valued section along paths; for this end one has to put $\Psi\colon \gamma \mapsto {}_\gamma\Psi,\ \gamma\colon J \to M$, with ${}_\gamma\Psi\colon x \mapsto \{\Psi_\gamma(t) | \gamma(t) = x,\ t \in J\}$ for $x \in \gamma(J)$. If one employs multiple-valued sections along paths, the basic problem is how exactly the values corresponding to some 'time' value t are chosen and how the transition between different 'time' values is depicted; of course, this problem arises at the points of self-intersections of γ, if any. Mathematically the work with multiple-valued maps is considerably more difficult than the treatment of single valued ones. The correct regorous treatment of Ψ as a section requires additional rules describing, besides the correspondence $\gamma(t) \mapsto {}_\gamma\Psi(\gamma(t))$, the mapping $t \mapsto \Psi_\gamma(t)$ which is equivalent

to the consideration of Ψ as a lifting of paths. By this reason, in the general case, we shall look on Ψ as a lifting of paths. There is one important special case when both approaches to Ψ are transparently equivalent: when only paths γ without self-intersections are employed. This is a consequence from the fact that now the map $\gamma\colon J \to \gamma(J)$ is bijective as $\gamma\colon J \to M$ is injective. In particular, if for given γ there is a subinterval $J' \subset J$ such that the restricted path $\gamma|_{J'}$ is injective, the maps $\Psi_{\gamma|_{J'}}$ and ${}_{\gamma|_{J'}}\Psi$ are completely equivalent representations of Ψ along $\gamma|J'$.

The physical preference to interpret Ψ as a section of lifting depends on the concrete choice of M and the corresponding interpretation of γ. For example, if M is the space-time of special or general relativity and γ is the world line of (real point-like) observer, then γ is without self-intersections and Ψ along γ can naturally be interpreted as section in $\mathrm{Sec}(F, \pi, M)|_{\gamma(J)}$. On the other hand, if $M = \mathbb{E}^3$ is the Euclidean space of classical mechanics and $\gamma\colon J \to \mathbb{E}^3$ is the trajectory of some point-like object, treated as an observer, then γ could have self-intersections and, correspondingly, Ψ is more easily treated as lifting of paths.

Definition 4.3. The unique lifting of paths Ψ or (multiple-valued) section along paths Ψ corresponding to the state vector ψ from conventional quantum mechanics will be called state lifting (of paths) or state section (along paths) respectively.

For brevity, we call, by abuse of the language, a particular value of Ψ, say $\Psi_\gamma(t)$, a *bundle state vector* (at a moment t, or, more precisely, at the (space) point $\gamma(t)$ and at the instant t, i.e. at a space-time point $(\gamma(t), t)$ if M is treated as a space-time model).

Since the entering in (4.3) mappings l_x, $x \in M$, are isomorphisms, the correspondences

$$\begin{aligned}\text{STATE VECTOR} &\iff \text{STATE LIFTING OF PATHS}\\ &\iff \text{STATE SECTION ALONG PATHS}\end{aligned} \tag{4.4}$$

are bijective (isomorphisms).[28] Hence, the description of a quantum system via state vectors, or liftings of paths, or sections along paths are equivalent.

On the base of postulates 4.1 and 4.2, the formalism of conventional quantum mechanics concerning solely the wave function (state vector) ψ can be transferred, equivalently, in the Hilbert bundle description in terms of the state lifting Ψ of paths. Equation (4.3) plays the major role in this reformulation. In this direction, our first goal is the Hilbert bundle description of the quantum evolution, i.e. the change of the state liftings/sections Ψ in time. In Sect. 5 we shall see that the bundle evolution of a quantum

[28] In (4.4) the state sections are, generally, multiple-values sections along paths.

system is represented by a suitable linear transport along paths in the system's Hilbert bundle. In the next, second, part of the present investigation the corresponding (bundle) equations of motion will be derived.

We completely understand that even at this early, introductory, stage of our work, a lot of concrete problems arise. They are connected with the transferring and/or interpretation of particular results of conventional quantum mechanics in the case of its bundle (re)formulation. These questions are, as a rule, out of the subject of the present work, devoted to the general formalism, and have to be considered separately of it. Regardless of this, we want to pay attention to one such problem which may lead to methodological difficulties.

It is well known [7–9], in most situations, the wave function (vector) of a particle is not localized at a single space point, but it is spread over some space region that could be even the entire space, as in the case of momentum eigenstate. *Prima facie* a superficial conclusion can be made that such a state is included in the 'local' Hilbert space at some point, i.e. in the fibre over it. Such a conclusion is generally entirely wrong (unless we are dealing with a state localized at a single point or the base M consists of a single point)! Suppose $\psi \in \mathcal{F}$ is the wave function of some quantum system with respect to some observer and $\psi(x,t)$ is its value at a space point x at time $t \in J$. Take the particular choice $M = \mathbb{R}^3$ and let $\gamma\colon J \to \mathbb{R}^3$ be the observer's trajectory.[29] Since the mappings l_x, $x \in M$ are (isometric) isomorphisms, from (4.3) follows that the bundle state vector $\Psi_\gamma(t)$ is non-zero if and only if the state vector $\psi(x,t)$ is non-zero. Let $W_t = \operatorname{Supp}\psi(\cdot,t) \subset \mathbb{R}^3 = M$ be the support of $\psi(\cdot,t)$, i.e. $\psi(x,t) \neq 0$ for $x \in W_t$ and $\psi(x,t) = 0$ for $x \notin W_t$ (if $W_t \neq M$). The above said implies $\Psi_\gamma(t) \neq 0$ iff $\gamma(t) \in W_t$, in other words $\Psi_\gamma(t) \neq 0$ iff $\pi\big(\Psi_\gamma(t)\big) \in W_t$. Consequently, the non-zero bundle state vectors are spread over the same region (of space) as the 'original' non-zero state vectors. Besides, the non-zero bundle state vectors are in the 'local' Hilbert spaces attached to the corresponding points in W_t, viz. $\Psi_\gamma(t) \neq 0$ is in the fibre $\pi^{-1}(\gamma(t)) \subset \pi^{-1}(W_t)$. In conclusion, the state liftings of paths are localized, i.e. are non-zero, in the same space region as the conventional wave functions. Analogous result can be obtained if we take for M other space(-time) models, such as V_4, M^4, etc.

Remark 4.2. The above interpretation of the case $M = \mathbb{R}^3$ (or $M = V_4$, etc.) is quite more natural than the one of conventional quantum mechanics: the non-zero values of the state liftings/sections are situated in the fibres just above the points at which the wave function is non-zero, while its values belong to an abstract Hilbert space which is highly non-local object, associated with the whole space(-time) rather than with some particular point in it. Mathematically our theory is valid if M is arbitrary manifold, but if M

[29] Other choices, such as $M = V_4, M^4, U^4, \ldots$, do not change anything in the next conclusions. The same concerns the interpretation of γ.

is not a space(-time) model, the above (and other) 'nice' interpretation(s) could be lost. For example, if M consists of a single point, $M = \{x\}$ we have $F = F_x = l_x^{-1}(\mathcal{F})$ and, according to note 4.4, we obtain an isomorphic copy in F of the standard quantum mechanics. Now, generally, for $\gamma\colon J \to \{x\}$ is hard to be found a 'good' interpretation but, if, e.g., x is in $\mathbb{R}^3$ (or in V_4, etc.), then γ can be interpreted as trajectory (world line) of an observer situated at a space point x during the whole period of 'observation'.

From one hand, as mentioned earlier, the postulate 4.1 and 4.2 are enough for the bundle reformulation of the state vector (wave function) formalism. In particular, the probabilistic interpretation of quantum mechanics is retained: since

$$\langle\psi(t)|\psi(t)\rangle = \langle\Psi_\gamma(t)|\Psi_\gamma(t)\rangle_{\gamma(t)}, \tag{4.5}$$

which is a corollary of (3.1) and (4.3)), the bundle state vector $\Psi_\gamma(t)$ can be interpreted as a probability amplitude. Form other hand, these postulates do not allow us to transfer in the bundle description the predictions of quantum mechanics concerning the observables. For this end, new initial assertions are required. They will be presented in Sect. 7 of this investigation in which the exploration of the observables in the bundle approach begins. In short, their essence is: in the bundle approach the observables are described via (Hermitian) liftings of paths or (multiple-valued) morphisms along paths (in the bundle of restricted morphisms of the system's Hilbert bundle or in the Hilbert bundle of states respectively) and their mean values (mathematical expectations) are such that they coincide with the mean values of the corresponding (Hermitian) operators representing the same observables in the Hilbert space quantum mechanics. On the ground of these assertions, the whole machinery of quantum mechanics (of pure states) can be reformulated in terms of fibre bundles. This will be done in the next sections of our work. Due to the just mentioned coincidence of the mean values of the operators and liftings corresponding to observables, *the predictions of Hilbert space and Hilbert bundle quantum mechanics are absolutely identical, i.e. these are different representations of a single theory, the quantum mechanics.* For the bundle description of mixed states, additional postulates are required. They will be presented further. As we shall see, in the bundle approach the mixed states are represented via density liftings of paths (or multiple-valued density sections along paths) such that the mean values of the liftings (or sections) corresponding to observables coincide with the mean values of the corresponding to them Hermitian operators (in the Hilbert space description) computed by means of the ordinary density operator (matrix). Consequently, as in the case of pure states, now we have also a complete coincidence of the predictions of Hilbert space and Hilbert bundle versions of quantum mechanics.

Beginning with the next section, following the above lines, the purpose

of this work is the bundle formulation of the general formalism of quantum mechanics.

4.4. Preliminary recapitulation

The summary and discussion of the bundle version of quantum mechanics will be presented in the concluding part of this work. Below we give a short abstract of them with the hope that it will help for the better understanding of our investigation. It also serves as a partial motivation for the present work.

The bundle formulation of quantum mechanics is a purely geometrical version of conventional quantum mechanics to which it is completely equivalent; hence these are simply different 'faces' of a single theory, the quantum mechanics. The proposed geometric formulation of quantum mechanics is *dynamical* in a sense that all geometrical structures employed for the description of a quantum system depend on and are determined form the dynamical characteristics of the system. The new form of the theory has three free parameters: the bundle's base M, the set $\{l_x|x \in M$ of point trivializing isometric isomorphisms, and the path $\gamma\colon J \to M$. The choice of these objects is external to quantum mechanics and is subjected to reasons like the physical interpretation of the theory and its connection with other physical theories. As a working hypothesis, we suggest to interpret M as a space(-time) model and γ as a trajectory (world line) of an observer along which the quantum evolution is studied.

In the Hilbert bundle description the system Hilbert space is replaced with a suitable Hilbert bundle. In it the system state is represented via appropriate state lifting of paths in the case of pure states or density liftings of paths if the state is mixed. In both cases, the quantum evolution in time is characterized by a linear transport along γ of the state lifting or density liftings in the system Hilbert bundle or in the bundle of its point-restricted morphisms over the base respectively. The corresponding equations of motion are derived. The probabilistic interpretation of quantum theory remains valid.

In the new bundle approach, the observables are describe via liftings of paths in the bundle of restricted morphisms over the base in the Hilbert bundle of states. They are so-defined that their mean values coincide with the mean values of the corresponding Hermitian operators representing observables in conventional quantum theory. Therefore the physical predictions of the Hilbert space and Hilbert bundle versions of quantum mechanics are identical. The bundle equations of motion, governing the time evolution of observables are derived.

From bundle's view-point, an observable is integral of motion iff it is a constant lifting of paths, viz. iff it is linearly transported in the bundle of restricted morphisms over the base in the Hilbert bundle of states with respect

to the linear transport induced in this bundle by the evolution transport of the state liftings.

We also pay attention to the bundle version of the different pictures of motion. The corresponding equations of motion for the state liftings (or density liftings) and observables are considered in the bundle pictures of motion. We point to an interesting result: in terms of local frames, the bundle Heisenberg picture of motion corresponds to the choice of a suitable normal frame, i.e. a frame in which the matrix of the evolution transport of the state liftings is unit. Since the normal frames are the mathematical objects corresponding to the physical concept of an inertial frame, the above means that the (bundle) Heisenberg picture of quantum mechanics is something like a 'quantum mechanics in a (bundle) inertial frame'.

At last, we consider problems concerning the role of observers, physical interpretation, and possible generalizations of bundle quantum mechanics. In these directions the new form of the theory admits a lot of developments which is due to the afore mentioned three free parameters in it. We point that the presented formalism can be transferred in the relativistic region too.

5. The (bundle) evolution transport

The purpose of this section is the Hilbert bundle description of the quantum evolution of a quantum system. More precisely, we want to find the time-dependence of the state liftings of paths (or sections along paths) of a system provided the time-dependence of its (conventional) wave function (state vector) is known.[30] We shall prove that this is achieved via a suitable linear transport along paths, called evolution transport, in the system's Hilbert bundle.

According to postulate 4.1, assertion (iii), the evolution of a system in the fibre $\mathcal{F}$ of the system's Hilbert bundle (F, π, M) is given via the evolution operator $\mathcal{U}$ (see Sect. 2). This operator has a 'transport like' properties, similar to (3.19)–(3.21). Indeed, using (2.1), we get $\psi(t_3) = \mathcal{U}(t_3, t_2)\psi(t_2) = \mathcal{U}(t_3, t_2)[\mathcal{U}(t_2, t_1)\psi(t_1)]$, $\psi(t_3) = \mathcal{U}(t_3, t_1)\psi(t_1)$, and $\psi(t_1) = \mathcal{U}(t_1, t_1)\psi(t_1)$ for every moments t_1, t_2, t_3 and arbitrary state vector ψ. Hence

$$\mathcal{U}(t_3, t_1) = \mathcal{U}(t_3, t_2) \circ \mathcal{U}(t_2, t_1), \tag{5.1}$$

$$\mathcal{U}(t_1, t_1) = \mathsf{id}_{\mathcal{F}}. \tag{5.2}$$

[30] The corresponding bundle equations of motion will be derived in the second part of this investigation.

Besides, by definition, $\mathcal{U}(t_2,t_1)\colon \mathcal{F}\to\mathcal{F}$ is a linear unitary operator, i.e. for $\lambda_i\in\mathbb{C}$ and $\psi_i(t_1)\in\mathcal{F}$, $i=1,2$, we have:

$$\mathcal{U}(t_2,t_1)\Big(\sum_{i=1,2}\lambda_i\psi_i(t_1)\Big)=\sum_{i=1,2}\lambda_i\mathcal{U}(t_2,t_1)\psi_i(t_1), \tag{5.3}$$

$$\mathcal{U}^\dagger(t_1,t_2)=\mathcal{U}^{-1}(t_2,t_1). \tag{5.4}$$

From (5.1) and (5.2), evidently, follows

$$\mathcal{U}^{-1}(t_2,t_1)=\mathcal{U}(t_1,t_2) \tag{5.5}$$

and consequently

$$\mathcal{U}^\dagger(t_1,t_2)=\mathcal{U}(t_1,t_2). \tag{5.6}$$

If one takes as a primary object the Hamiltonian $\mathcal{H}$, these facts are direct consequences of (2.10).

Thus the properties of the evolution operator are very similar to the ones defining a ((flat) Hermitian) linear transport along paths in a Hilbert bundle. In fact, below we show that the bundle analogue of the evolution operator is a kind of such transport.

Along any path γ, we define the bundle analogue of the evolution operator $\mathcal{U}(t,s)\colon \mathcal{F}\to\mathcal{F}$ as a linear mapping $U_\gamma(t,s)\colon F_{\gamma(s)}\to F_{\gamma(t)},\ s,t\in J$ such that

$$\Psi_\gamma(t)=U_\gamma(t,s)\Psi_\gamma(s) \tag{5.7}$$

for every instants of time $s,t\in J$. Hence U_γ connects the different time values of the bundle state vectors. Analogously to (5.1) and (5.2), now we have:

$$U_\gamma(t_3,t_1)=U_\gamma(t_3,t_2)\circ U_\gamma(t_2,t_1),\qquad t_1,t_2,t_3\in J, \tag{5.8}$$

$$U_\gamma(t,t)=\mathsf{id}_{F_{\gamma(t)}},\qquad t\in J. \tag{5.9}$$

Comparing (5.7) with (2.1) and using (4.3), we find

$$U_\gamma(t,s)=l^{-1}_{\gamma(t)}\circ\mathcal{U}(t,s)\circ l_{\gamma(s)},\qquad s,t\in J \tag{5.10}$$

or

$$\mathcal{U}(t,s)=l_{\gamma(t)}\circ U_\gamma(t,s)\circ l^{-1}_{\gamma(s)},\qquad s,t\in J. \tag{5.11}$$

This shows the equivalence of the description of evolution of a quantum systems via $\mathcal{U}$ and U_γ.

A trivial corollary of (5.10) is the *linearity* of U_γ and

$$U_\gamma^{-1}(t,s) = U_\gamma(s,t). \tag{5.12}$$

As $l_x\colon F_x \to \mathcal{F},\ x \in M$ are linear isomorphisms, from (5.8)–(5.10) follows that $U\colon \gamma \mapsto U_\gamma$ with $U_\gamma\colon (s,t) \mapsto U_\gamma(s,t) =: U^\gamma_{t\to s}\colon F_{\gamma(t)} \to F_{\gamma(s)}$ is a *linear transport along paths* in (F,π,M).[31] This transport is *Hermitian* (see Sect. 3). In fact, applying (3.8) to $U_\gamma(t,s)$ and using (5.10), we get

$$U^\ddagger_\gamma(t,s) = l^{-1}_{\gamma(t)} \circ \mathcal{U}^\dagger(s,t) \circ l_{\gamma(t)}. \tag{5.13}$$

So, using (5.6), once again (5.10), and (5.5), we find

$$U^\ddagger_\gamma(t,s) = U_\gamma(t,s) = U_\gamma^{-1}(s,t). \tag{5.14}$$

Hence $U_\gamma(t,s)$ is simultaneously Hermitian and unitary operator, as it should be for any Hermitian or unitary transport along paths in a Hilbert bundle (see Sect. 3). Consequently, U is an isometric transport along paths.

Above we defined the transport U by (5.7) from which (5.7)–(5.14) follow. It is a simple exercise to prove that if U is defined via (5.10), the remaining equations of (5.7)–(5.14) are fulfilled. Consequently, (5.7) and (5.10) are equivalent definitions of the transport U along paths.

Definition 5.1. The isometric linear transport U along paths, defined via equation (5.7) or (5.10), in the system's Hilbert bundle (F,π,M) of states is called evolution transport (of the system) or bundle evolution operator.

In this way, we see that the evolution transport U is a Hermitian (and hence unitary) linear transport along paths in (F,π,M). Consequently, to any unitary evolution operator $\mathcal{U}$ in the Hilbert space $\mathcal{F}$ there corresponds a unique isometric linear transport U along paths, the evolution transport, in the Hilbert bundle (F,π,M) and vice versa.

Let us summarize. In the Hilbert bundle description, the time evolution of a quantum system is represented by means of the evolution transport along paths in the system's Hilbert bundle. It connects the different time values of the state liftings according to (5.7) along the reference path γ. Equation (5.10) is the link between the evolution transport and evolution operator; it is equivalent to (4.3) provided (5.7) is postulated.

[31] In the context of quantum mechanics it is more natural to define $U_\gamma(s,t)$ from $F_{\gamma(t)}$ into $F_{\gamma(s)}$ instead from $F_{\gamma(s)}$ into $F_{\gamma(t)}$, as is the map $U^\gamma_{s\to t} = U_\gamma(t,s)\colon F_{\gamma(s)} \to F_{\gamma(t)}$. The latter notation is better in the general theory of transports along paths [31, 32]. Consequently, when applying results from [31, 32], we have to remember that they are valid for the maps $U^\gamma_{s\to t}$ (or $U^\gamma\colon (s,t) \mapsto U^\gamma_{s\to t}$). That is why for the usage of some results concerning general linear transports along paths from [31,32] for $U_\gamma(s,t)$ or U_γ one has to write them for $U^\gamma_{s\to t}$ (or U^γ) and then to use the connection $U^\gamma_{s\to t} = U_\gamma(t,s) = U_\gamma^{-1}(s,t)$ (or $U^\gamma = U_\gamma^{-1}$). Some results for $U^\gamma_{s\to t}$ and $U_\gamma(s,t)$ coincide but this is not always the case. In short, the results for linear transports along paths are transferred to the considered in this work case by replacing $L^\gamma_{s\to t}$ with $U_\gamma(t,s) = U_\gamma^{-1}(s,t)$.

6. The bundle equations of motion

In conventional quantum mechanics, the time-dependence of the state vector $\psi \in \mathcal{F}$ of a quantum system is governed via the Schrödinger equation (2.6). It is natural to expect the existence of an analogous equation for the state lifting Ψ replacing ψ by (4.3) in the bundle description of quantum mechanics. The derivation of this equation (or of its variants), which should be only in bundle terms, is the major purpose of the present section. Regardless of some technical problems, the idea is quite simple: using (4.3) and (5.7) or (5.10), one should transform the Schrödinger equation in 'pure' bundle terms. A realization of such a procedure is given below. The resulting (invariant) bundle equation of motion has an amazingly transparent geometrical meaning: it expresses the fact that the state liftings/sections are linearly transported along the reference path along which the quantum evolution is explored.

6.1. Derivation of the equations

If we substitute (5.11) into (2.6)–(2.10), we 'get' the 'bundle' analogues of (2.6)–(2.10). But they will be wrong! This is due to the fact that they will contain partial derivatives like $\partial l_{\gamma(t)}/\partial t$, $\partial \Psi_\gamma(t)/\partial t$, and $\partial U_\gamma(t,t_0)/\partial t$, which are not defined at all. For instance, for in the first case we must have $\partial l_{\gamma(t)}/\partial t = \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}(l_{\gamma(t+\varepsilon)} - l_{\gamma(t)})\right)$, but the 'difference' in this limit is not defined (for $\varepsilon \neq 0$) because $l_{\gamma(t+\varepsilon)}$ and $l_{\gamma(t)}$ act on different spaces, viz. on $F_{\gamma(t+\varepsilon)}$ and $F_{\gamma(t)}$ respectively. The same is the situation with $\partial U_\gamma(t,t_0)/\partial t$. The most obvious is the contradiction in the following relation $\partial\Psi_\gamma(t)/\partial t = \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}(\Psi_\gamma(t+\varepsilon) - \Psi_\gamma(t))\right)$, because $\Psi_\gamma(t+\varepsilon)$ and $\Psi_\gamma(t)$ belong to different (for $\varepsilon \neq 0$) vector spaces.

One can go through this difficulty by defining, for example, $\partial\Psi_\gamma(t)/\partial t$ like $l^{-1}_{\gamma(t)}\partial\psi_\gamma(t)/\partial t$ (cf. (4.1)) but this does not lead to some important and new results.

To overcome this problem, we are going to introduce local bases (or coordinates) and to work with the matrices of the corresponding operators and vectors in them.

Let $\{e_a(x), \quad a \in \Lambda\}$ be a basis in $F_x = \pi^{-1}(x)$, $x \in M$. The indices $a, b, c, \ldots \in \Lambda$ may take discrete, or continuous, or both values. More precisely, the set Λ has a decomposition $\Lambda = \Lambda_d \bigcup \Lambda_c$ where Λ_d is a union of (a finite or countable) subsets of $\mathbb{N}$ (or, equivalently, of $\mathbb{Z}$) and Λ_c is union of subsets of $\mathbb{R}$ (or, equivalently, of $\mathbb{C}$). Note that Λ_d or Λ_c, but not both, can be empty. This is why sums like[32] $\lambda^a e_a(x)$ or $\lambda_a\mu^a$ for $a \in \Lambda$ and $\lambda^a, \mu_a \in \mathbb{C}$ must be understood as a sum over the discrete

[32]Here and henceforth in this work, we use the Einstein rule for summation over indices repeated on different levels.

(enumerable) part(s) of Λ, if any, plus the (Stieltjes or Lebesgue) integrals over the continuous part(s) of Λ, if any. For instance: $\lambda^a e_a(x) := \sum_{a\in\Lambda}\lambda^a e_a(x) := \sum_{a\in\Lambda_d}\lambda^a e_a(x) + \int_{a\in\Lambda_c}\lambda^a e_a(x)da$. By this reason it is better to write $\sum\!\!\!\!\!\!\int_{a\in\Lambda} := \sum_{a\in\Lambda_d} + \int_{a\in\Lambda_c} da$ instead of $\sum_{a\in\Lambda}$, but we shall avoid this complicated notation by using the assumed summation convention on indices repeated on different levels.[33]

The matrices corresponding to vectors or operators in a given field of bases will be denoted with the same symbol but in **boldface**, for example: $\boldsymbol{U}_\gamma(t,s) := \left[(U_\gamma(t,s))^a{}_b\right]$ and $\boldsymbol{\Psi}_\gamma(s) := \left[\Psi^a_\gamma(s)\right]$, where $U_\gamma(t,s)\,(e_b(\gamma(s))) =: (U_\gamma(t,s))^a{}_b\, e_a(\gamma(t))$ and $\Psi_\gamma(s) =: \Psi^a_\gamma(s)e_a(\gamma(s))$.[34]

Analogously, we suppose in $\mathcal{F}$ to be fixed a basis $\{f_a(t), \quad a\in\Lambda\}$ with respect to which we shall use the same bold-faced matrix notation, for instance: $\boldsymbol{\mathcal{U}}(t,s) = \left[\mathcal{U}^b{}_a(t,s)\right]$, $\mathcal{U}(t,s)\,(f_a(s)) =: (\mathcal{U}(t,s))^b{}_a\, f_b(t)$, $\boldsymbol{\psi}(t) = [\psi^a(t)]$, $\psi(t) =: \psi^a(t)f_a(t)$ and the 'two-point' matrix $\boldsymbol{l}_x(t) = \left[(l_x)^b{}_a(t)\right]$ is defined via $l_x\,(e_a(x)) =: (l_x)^b{}_a(t)f_b(t)$. Generally $\boldsymbol{l}_x(t)$ depends on x and t, but if $x=\gamma(s)$ for some $s\in J$, we put $t=s$ as from physical reasons is clear that $F_{\gamma(t)}$ corresponds to $\mathcal{F}$ at the 'moment' t, i.e. the components of $l_{\gamma(s)}$ are with respect to $\{e_a(\gamma(s))\}$ and $\{f_a(s)\}$. The same remark concerns 'two-point' objects like $U_\gamma(t,s)$ and $\mathcal{U}(t,s)$ whose components will be taken with respect to pairs of bases like $(\{e_a(\gamma(t))\},\{e_a(\gamma(s))\})$ and $(\{f_a(t)\},\{f_a(s)\})$ respectively.

Evidently, the equations (4.1), (5.7)–(5.10) remain valid *mutatis mutandis* in the introduced matrix notation: the kernel letters have to be made bold-faced, the operator composition (product) must be replaced by matrix multiplication, and the identity map id_{F_x} has to be replaced by the unit matrix $\mathbb{1}_{F_x} := \left[\delta^b_a\right] := \left[(\mathsf{id}_{F_x})^b{}_a\right]$ of F_x in $\{e_a(x)\}$. Here $\delta^b_a = 1$ for $a=b$ and $\delta^b_a = 0$ for $a\neq b$, which means that $e_a(x) = \delta^b_a e_b(x)$. For instance, using the above definitions, one verifies that (5.10) is equivalent to

$$\boldsymbol{U}_\gamma(t,s) = \boldsymbol{l}^{-1}_{\gamma(t)}(t)\boldsymbol{\mathcal{U}}(t,s)\boldsymbol{l}_{\gamma(s)}(s). \tag{6.1}$$

Let $\boldsymbol{\Omega}(x) := \left[\Omega_a{}^b(x)\right]$ and $\boldsymbol{\omega}(t) := \left[\omega_a{}^b(t)\right]$ be nondegenerate matrices. The changes

$$\{e_a(x)\}\to\{e'_a(x) := \Omega_a{}^b(x)e_b(x)\}, \qquad \{f_a(t)\}\to\{e'_a(t) := \omega_a{}^b(t)e_b(t)\}$$

of the bases in F_x and $\mathcal{F}$, respectively, lead to the transformation of the

[33] For details concerning infinite dimensional matrices see, for instance, [38] and [9, chapter VII, § 18]. A comprehensive presentation of the theory of infinite matrices is given in [61]; this book is mainly devoted to infinite discrete matrices but it contains also some results on continuous infinite matrices related to Hilbert spaces.

[34] The matrices $\boldsymbol{\mathcal{U}}(t,s)$ and $\boldsymbol{U}(t,s)$ are closely related to propagator functions [42], but we will not need these explicit connections. For explicit calculations and construction of (t,s), see [42, § 21, §22]

matrices of the components of $\Phi_x \in F_x$ and $\phi \in \mathcal{F}$, according to

$$\boldsymbol{\Phi}_x \mapsto \boldsymbol{\Phi}'_x = \big(\boldsymbol{\Omega}^\top(x)\big)^{-1}\boldsymbol{\Phi}_x, \quad \boldsymbol{\phi} \mapsto \boldsymbol{\phi}' = \big(\boldsymbol{\omega}^\top(t)\big)^{-1}\boldsymbol{\phi}. \tag{6.2}$$

Here the super script $\top$ means matrix transposition, for example $\boldsymbol{\Omega}^\top(x) := \big[\big(\Omega^\top(x)\big)^a{}_b\big]$ with $\big(\Omega^\top(x)\big)^a{}_b := \Omega_b{}^a(x)$. One easily verifies the transformation

$$\boldsymbol{l}_x(t) \mapsto \boldsymbol{l}'_x(t) = \big(\boldsymbol{\omega}^\top(t)\big)^{-1}\boldsymbol{l}_x(t)\boldsymbol{\Omega}^\top(x) \tag{6.3}$$

of the components of the linear isomorphisms $l_x\colon F_x \to \mathcal{F}$ under the above changes.

For any operator $\mathcal{A}(t)\colon \mathcal{F} \to \mathcal{F}$ we have

$$\boldsymbol{\mathcal{A}}(t) \mapsto \boldsymbol{\mathcal{A}}'(t) = \big(\boldsymbol{\omega}^\top(t)\big)^{-1}\boldsymbol{\mathcal{A}}(t)\boldsymbol{\omega}^\top(t). \tag{6.4}$$

Analogously, if $A(t)$ is a morphism of (F, π, M), i.e. if $A\colon F \to F$ and $\pi \circ A = \mathsf{id}_M$, and $A_x := A(t)|_{F_x}$, then

$$\boldsymbol{A}_x(t) \mapsto \boldsymbol{A}'_x(t) = \big(\boldsymbol{\Omega}^\top(t)\big)^{-1}\boldsymbol{A}_x(t)\boldsymbol{\Omega}^\top(t). \tag{6.5}$$

Note that the components of $\mathcal{U}(t,s)$, when referred to a pair of bases $\{e_a(t)\}$ and $\{e_a(s)\}$, transform according to

$$\boldsymbol{\mathcal{U}}(t,s) \mapsto \boldsymbol{\mathcal{U}}'(t,s) = \big(\boldsymbol{\omega}^\top(t)\big)^{-1}\boldsymbol{\mathcal{U}}(t,s)\boldsymbol{\omega}^\top(s). \tag{6.6}$$

Analogously, the change $\{e_a(\gamma(t))\} \to \{e'_a(t;\gamma) := \Omega_a{}^b(t;\gamma)e_b(\gamma(t))\}$, with a nondegenerate matrix $\boldsymbol{\Omega}(t;\gamma) := \big[\Omega_a{}^b(t;\gamma)\big]$ along γ, implies [35]

$$\boldsymbol{U}_\gamma(t,s) \mapsto \boldsymbol{U}'_\gamma(t,s) = \big(\boldsymbol{\Omega}^\top(t;\gamma)\big)^{-1}\boldsymbol{U}_\gamma(t,s)\boldsymbol{\Omega}^\top(s;\gamma). \tag{6.7}$$

Substituting $\psi(t) = \psi^a(t)f_a(t)$ into (2.6), we get the *matrix Schrödinger equation*

$$\frac{d\boldsymbol{\psi}(t)}{dt} = \boldsymbol{\mathcal{H}}^{\mathbf{m}}(t)\boldsymbol{\psi}(t) \tag{6.8}$$

where

$$\boldsymbol{\mathcal{H}}^{\mathbf{m}}(t) := \boldsymbol{\mathcal{H}}(t) - \mathrm{i}\hbar\boldsymbol{E}(t) \tag{6.9}$$

is the *matrix Hamiltonian* (in the Hilbert space description). Here $\boldsymbol{E}(t) = \big[E_a{}^b(t)\big]$ determines the expansion of $df_a(t)/dt$ over $\{f_a(t)\} \subset \mathcal{F}$, that is $df_a(t)/dt = E_a{}^b(t)f_b(t)$; if $f_a(t)$ are independent of t, which is the usual case, we have $\boldsymbol{E}(t) = 0$. In the last case $\boldsymbol{\mathcal{H}}^{\mathbf{m}} = \boldsymbol{\mathcal{H}}$. It is important to be noted that $\boldsymbol{\mathcal{H}}^{\mathbf{m}}$ is independent of $\boldsymbol{E}(t)$. In fact, applying (2.9) to

[35] Cf. [31, equation (2.11)] or [32, equation (4.10)], where the notation $H(t,s;\gamma) = \boldsymbol{U}_\gamma(s,t;\gamma)$ and $A(t) = \boldsymbol{\Omega}^\top(t;\gamma)$ is used.

the basic vector $f_a(t)$, we get $\mathcal{H}(t)f_a(t) = \mathrm{i}\hbar[(\frac{\partial}{\partial t}\mathcal{U}(t,t_0))f_b(t_0)]\mathcal{U}_a{}^b(t_0,t) = \mathrm{i}\hbar[\frac{\partial}{\partial t}(f_c(t)\mathcal{U}_b{}^c(t,t_0))]\mathcal{U}_a{}^b(t_0,t)$, so that

$$\boldsymbol{\mathcal{H}}(t) = \mathrm{i}\hbar\frac{\partial\boldsymbol{\mathcal{U}}(t,t_0)}{\partial t}\boldsymbol{\mathcal{U}}(t_0,t) + \mathrm{i}\hbar\boldsymbol{E}(t) \tag{6.10}$$

which leads to

$$\boldsymbol{\mathcal{H}}^{\mathbf{m}}(t) = \mathrm{i}\hbar\frac{\partial\boldsymbol{\mathcal{U}}(t,t_0)}{\partial t}\boldsymbol{\mathcal{U}}(t_0,t). \tag{6.11}$$

Substituting the matrix form of (4.1) into (6.8), we find the *matrix-bundle Schrödinger equation*

$$\mathrm{i}\hbar\frac{d\boldsymbol{\Psi}_\gamma(t)}{dt} = \boldsymbol{H}^{\mathbf{m}}_\gamma(t)\boldsymbol{\Psi}_\gamma(t) \tag{6.12}$$

where the *matrix-bundle Hamiltonian* is

$$\boldsymbol{H}^{\mathbf{m}}_\gamma(t) := \boldsymbol{l}^{-1}_{\gamma(t)}(t)\boldsymbol{\mathcal{H}}(t)\boldsymbol{l}_{\gamma(t)}(t) - \mathrm{i}\hbar\boldsymbol{l}^{-1}_{\gamma(t)}(t)\left(\frac{d\boldsymbol{l}_{\gamma(t)}(t)}{dt} + \boldsymbol{E}(t)\boldsymbol{l}_{\gamma(t)}(t)\right). \tag{6.13}$$

Combining (6.9) and (6.13), we find the following connection between the conventional and bundle matrix Hamiltonians:

$$\boldsymbol{H}^{\mathbf{m}}_\gamma(t) = \boldsymbol{l}^{-1}_{\gamma(t)}(t)\boldsymbol{\mathcal{H}}^{\mathbf{m}}(t)\boldsymbol{l}_{\gamma(t)}(t) - \mathrm{i}\hbar\boldsymbol{l}^{-1}_{\gamma(t)}(t)\frac{d\boldsymbol{l}_{\gamma(t)}(t)}{dt}. \tag{6.14}$$

Remark 6.1. Choosing $e_a(x) = l_x^{-1}(f_a)$ for $df_a(t)/dt = 0$, we get $\boldsymbol{l}_x(t) = \left[\delta^b_a\right]$. Then $\boldsymbol{H}_\gamma(t) = \boldsymbol{\mathcal{H}}(t)$. So, as $\mathcal{H}^\dagger = \mathcal{H}$, we have $\left(\boldsymbol{H}^{\mathbf{m}}_\gamma(t)\right)^\dagger = \boldsymbol{\mathcal{H}}^\dagger(t) = \boldsymbol{\mathcal{H}}(t) = \boldsymbol{H}^{\mathbf{m}}_\gamma(t)$ where we use the dagger ($\dagger$) to denote also matrix Hermitian conjugation. Here $\boldsymbol{H}^{\mathbf{m}}_\gamma(t)$ is a Hermitian matrix in the chosen basis, but in other bases it may not be such (see below (6.24)). Analogously, choosing $\{f_a(t)\}$ such that $\boldsymbol{E}(t) = 0$, we see that $\boldsymbol{\mathcal{H}}^{\mathbf{m}}(t) = \boldsymbol{\mathcal{H}}(t)$ is a Hermitian matrix, otherwise it may not be such.

Remark 6.2. Note, due to (6.14), the transition $\boldsymbol{\mathcal{H}}^{\mathbf{m}} \to \boldsymbol{H}^{\mathbf{m}}_\gamma$ is very much alike a gauge (or connection) transformation [14] (see also below equations (6.22)–(6.24)).

Because of (6.12) and (5.7) there is a bijective correspondence between $\boldsymbol{U}_\gamma$ and $\boldsymbol{H}^{\mathbf{m}}_\gamma$ expressed through the initial-value problem (cf. (2.8))

$$\mathrm{i}\hbar\frac{\partial\boldsymbol{U}_\gamma(t,t_0)}{\partial t} = \boldsymbol{H}^{\mathbf{m}}_\gamma(t)\,\boldsymbol{U}_\gamma(t,t_0), \qquad \boldsymbol{U}_\gamma(t_0,t_0) = \mathbb{1}_{F_{\gamma(t_0)}}, \tag{6.15}$$

or via the equivalent to it integral equation

$$\boldsymbol{U}_\gamma(t,t_0) = \mathbb{1}_{F_{\gamma(t_0)}} + \frac{1}{\mathrm{i}\hbar}\int\limits_{t_0}^{t}\boldsymbol{H}^{\mathbf{m}}_\gamma(\tau)\,\boldsymbol{U}_\gamma(\tau,t_0)d\tau. \tag{6.16}$$

So, if $\boldsymbol{H}^{\mathbf{m}}_{\gamma}$ is given, we have (cf. (2.10))

$$\boldsymbol{U}_{\gamma}(t,t_0)=\operatorname{Texp}\int\limits_{t_0}^{t}\frac{1}{\mathrm{i}\hbar}\boldsymbol{H}^{\mathbf{m}}_{\gamma}(\tau)d\tau \tag{6.17}$$

and, conversely, if $\boldsymbol{U}_{\gamma}$ is given, then (cf. (2.9) and (6.11)) [36]

$$\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t)=\mathrm{i}\hbar\frac{\partial\boldsymbol{U}_{\gamma}(t,t_0)}{\partial t}\boldsymbol{U}^{-1}_{\gamma}(t,t_0)=\frac{\partial\boldsymbol{U}_{\gamma}(t,t_0)}{\partial t}\boldsymbol{U}_{\gamma}(t_0,t). \tag{6.18}$$

The next step is to write the above matrix equations into an invariant, i.e. basis-independent, form. For this purpose we shall use the introduce in [31, 32] derivation along paths uniquely corresponding to any linear transport along paths in a vector bundle.

According to definitions 3.3 and 3.4 the derivation along paths corresponding to the bundle evolution transport U is a linear mapping

$$D\colon \operatorname{PLift}^1(F,\pi,M)\to\operatorname{PLift}^0(F,\pi,M),$$

$\operatorname{PLift}^k(F,\pi,M)$ being the set of C^k liftings of paths from M to F, such that for every C^1 lifting λ of paths and every path $\gamma\colon J\to M$, we have $D\colon\lambda\mapsto D(\lambda)=D\lambda$ and $D\lambda\colon\gamma\mapsto D^{\gamma}(\lambda)=(D\lambda)_{\gamma}$ is defined by $D^{\gamma}\lambda\colon s\mapsto D^{\gamma}_{s}\lambda\in F_{\gamma(s)}$ with

$$D^{\gamma}_{s}(\lambda):=\lim_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon}\big[U_{\gamma}(s,s+\varepsilon)\lambda_{\gamma}(s+\varepsilon)-\lambda_{\gamma}(s)\big]\Big\} \tag{6.19}$$

where $\lambda\colon\gamma\mapsto\lambda_{\gamma}$.

By (3.42) (see also [31, equation (2.27)] or [32, proposition 4.2]) the explicit local form of (6.19) in a frame $\{e_i(\cdot,\gamma)\}$ along γ is

$$D^{\gamma}_{s}\lambda=\left(\frac{d\lambda^{a}_{\gamma}(s)}{ds}+\Gamma^{a}{}_{b}(s;\gamma)\lambda^{b}_{\gamma}(s)\right)e_a(s;\gamma) \tag{6.20}$$

where the *coefficients* $\Gamma^{b}{}_{a}(s;\gamma)$ of U are defined by (cf. (3.43))

$$\Gamma^{b}{}_{a}(s;\gamma):=\left.\frac{\partial\left(U_{\gamma}(s,t)\right)^{b}{}_{a}}{\partial t}\right|_{t=s}=-\left.\frac{\partial\left(U_{\gamma}(t,s)\right)^{b}{}_{a}}{\partial t}\right|_{t=s}. \tag{6.21}$$

Using (5.9) and (6.18), both for $t_0=t$, we see that

$$\boldsymbol{\Gamma}_{\gamma}(t):=\big[\Gamma^{b}{}_{a}(t;\gamma)\big]=-\frac{1}{\mathrm{i}\hbar}\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t) \tag{6.22}$$

[36] Expressions like $(\partial\mathcal{U}(t,t_0)/\partial t)\mathcal{U}(t_0,t)$, $(\partial\boldsymbol{U}_{\gamma}(t,t_0)/\partial t)\boldsymbol{U}^{-1}_{\gamma}(t,t_0)$, and $\mathcal{U}(t,t_0)\mathcal{U}(t_0,t_1)$ are independent of t_0 due to [31, propositions 2.1 and 2.4] or [32, propositions 2.1 and 2.4] (see also (3.23), (3.44), and [37, lemma 3.1]).

which expresses a fundamental result: *up to a constant the matrix-bundle Hamiltonian coincides with the matrix of coefficients of the bundle evolution transport* (in a given field of bases). Let us recall, using another arguments, analogous result was obtained in [1, sect. 5].

There are two invariant operators corresponding to the Hamiltonian $\mathcal{H}$ in $\mathcal{F}$: the evolution transport U and the corresponding to it derivation along paths D. Equations (6.12)–(6.22), as as well as the general results of [31, § 2] and [32, § 4], imply that these three operators, namely $\mathcal{H}$, U, and D, are equivalent in a sense that if one of them is given, then the remaining ones are uniquely determined.

Example 6.1. *Let $\{e_a(x)\}$ be fixed by $e_a(x) = l_x^{-1}(f_a)$ for $df(t)/dt = 0$. Then $\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t)$ is a Hermitian matrix (see remark 6.1). Consequently, in this case, $\boldsymbol{\Gamma}_{\gamma}(t)$ is anti-Hermitian, i.e. $(\boldsymbol{\Gamma}_{\gamma}(t))^{\dagger} = -\boldsymbol{\Gamma}_{\gamma}(t)$. Note that for other choices of the bases this property may not hold.*

Example 6.2. *Let $\mathcal{H}$ be given and independent of t, i.e. $\partial\mathcal{H}(t)/\partial t = 0$, and $\{e_a(x)\}$ be fixed by $e_a(x) = l_x^{-1}(f_a)$ for $df(t)/dt = 0$. Then $\boldsymbol{l}_x(t) = \left[\delta^b_a\right]$ with δ^b_a defined above. Equations (6.13) and (6.22) yield $\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t) = \boldsymbol{\mathcal{H}}(t)$ and $\boldsymbol{\Gamma}_{\gamma}(t) = -\boldsymbol{\mathcal{H}}(t)/\mathrm{i}\hbar$. Finally, now the solution of (6.15) is $\boldsymbol{U}_{\gamma}(t, t_0) = \exp\left(\boldsymbol{\mathcal{H}}(t)(t - t_0)/\mathrm{i}\hbar\right)$ (cf. (6.17)).*

According to [31, equation (2.30)] (or [32, equation (4.11)]) and footnote 31 on page 37, if the basis $\{e_a(t;\gamma)\}$ along γ is change to $\{e'_a(t;\gamma) = \Omega_a{}^b(t;\gamma)e_b(\gamma(t))\}$ with $\det\boldsymbol{\Omega}(t;\gamma) \neq 0$, $\boldsymbol{\Omega}(t;\gamma) := \left[\Omega_a{}^b(t;\gamma)\right]$, then $\boldsymbol{\Gamma}_{\gamma}(t)$ transforms into[37] (see (3.46))

$$\boldsymbol{\Gamma}'_{\gamma}(t) = (\boldsymbol{\Omega}^{\top}(t;\gamma))^{-1}\boldsymbol{\Gamma}_{\gamma}(t)\boldsymbol{\Omega}^{\top}(t;\gamma) + (\boldsymbol{\Omega}^{\top}(t;\gamma))^{-1}\frac{d\boldsymbol{\Omega}^{\top}(t;\gamma)}{dt}. \tag{6.23}$$

This result is also a corollary of (6.6) and (6.21).

Hence (see (6.22)), the matrix-bundle Hamiltonian undergoes the change $\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t) \mapsto {}'\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t)$ where

$${}'\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t) = (\boldsymbol{\Omega}^{\top}(t;\gamma))^{-1}\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t)\boldsymbol{\Omega}^{\top}(t;\gamma) - \mathrm{i}\hbar(\boldsymbol{\Omega}^{\top}(t;\gamma))^{-1}\frac{d\boldsymbol{\Omega}^{\top}(t;\gamma)}{dt}. \tag{6.24}$$

This result can be deduced from (6.14) too.

Now we are able to write into an invariant form the matrix-bundle Schrödinger equation (6.12). Substituting (6.22) into (6.12) and using (6.20), we find that (6.12) is equivalent to

$$D^{\gamma}_t\Psi = 0 \tag{6.25}$$

[37] In [31, 32] the matrix $A(t) = \boldsymbol{\Omega}^{\top}(t;\gamma)$ instead of $\boldsymbol{\Omega}(t;\gamma)$.

or, as $t \in J$ is arbitrary, to

$$D^{\gamma}\Psi = 0. \tag{6.26}$$

Since $\gamma\colon J \to M$ is arbitrary, the last equation can be rewritten as

$$D\Psi = 0. \tag{6.27}$$

This is the (invariant) *bundle Schrödinger equation* (for the state liftings). Since it coincides with the *linear transport equation* [62, definition 5.2] for the evolution transport, it has a very simple and fundamental geometrical meaning. By [62, proposition 5.4] this is equivalent to the statement that Ψ_{γ} *is a (linearly) transported along* γ *lifting* with respect to the evolution transport (expressed in other terms via (5.7); see [37, definition 2.2]). Note that (6.25) and (5.7) are compatible as [32, equation (4.5)] is fulfilled (see also [31, equation (2.25)]): $D^{\gamma}_t(\overline{U}) \equiv 0,\ t \in J$ where $\overline{U} \in \mathrm{PLift}(F, \pi, M)$ is the lifting of paths generated by U (see definition 3.5). Moreover, if D is given (independently of U, e.g. through (6.20)), from [62, proposition 5.4] follows that U is the unique solution of the (invariant) initial-value problem[38]

$$D^{\gamma}_t(\overline{U}) = 0 \qquad \overline{U}_{\gamma}(t_0, t_0) = \mathsf{id}_{F_{\gamma(t_0)}} \tag{6.28}$$

for fixed $t_0 \in J$. Since here $t \in J$ and $\gamma\colon J \to M$ are arbitrary, the equation in this initial-value problem is equivalent to

$$D^{\gamma}(\overline{U}) = 0 \tag{6.29}$$

or to

$$D(\overline{U}) = 0. \tag{6.30}$$

This is the *bundle Schrödinger equation for the evolution transport* U.

Remark 6.3. Mathematically equation (6.27) (or (6.25)) is a trivial corollary of (5.7) and (3.40). But this derivation of (6.27) leaves open the problem for its relation (equivalence) with the Schrödinger's one. Besides, such a 'quick' derivation of (6.27) leaves hidden the above-pointed properties of the matrix Hamiltonians, in particular the fundamental relation (6.22).

6.2. Inferences

Thus we see that there are two equivalent ways for describing the unitary evolution of a quantum system: (i) by means of the evolution transport $\mathcal{U}$ (see (2.1)) or by the Hermitian Hamiltonian $\mathcal{H}$ (see (2.6)) in the Hilbert space $\mathcal{F}$ (which is the typical fibre in the bundle description) and (ii) via

[38] In fact, (6.28) is the inversion of (6.19) with respect to U.

the evolution transport U (see (5.7)), which is a Hermitian (and unitary) transport along paths, or the derivation D along paths (see (6.25)) in the Hilbert bundle (F, π, M). In the bundle description U corresponds to $\mathcal{U}$ (see (5.10)) and D to $\mathcal{H}$ (see (6.20) and (6.22)).

We derived the bundle Schrödinger equation (6.27) from the 'classical' Schrödinger equation (2.6); the equivalence of the two equations is evident from the above considerations.

Now we have at our disposal all tools required for pure bundle description of the evolution of a quantum system.

Given a system characterized by a derivation D along paths. If the system's bundle state victor Ψ^0_γ is know along $\gamma\colon J \to M$ at a point $t_0 \in J$, the state lifting Ψ of paths is a solution of the bundle Schrödinger equation (6.25) under the initial condition

$$\Psi_\gamma(t_0) = \Psi^0_\gamma. \tag{6.31}$$

By virtue of (6.20), equation (6.25) and the condition (6.31) from a standard initial-value problem for a first order system of ordinary differential equations (with respect to the time t) which has solutions along γ [45].[39] This solution is

$$\Psi_\gamma(t) = U_\gamma(t, t_0)\Psi^0_\gamma$$

where the evolution transport U could be found as the unique solution of the initial-value problem (6.28)

Above we supposed the system to be described via a derivation D along paths instead by a Hamiltonian $\mathcal{H}$. These are equivalent approaches. Actually, in a local field of bases along γ, the matrix of $\mathcal{H}$ and the one of the coefficients of D are connected by (6.22) and (6.13) and, hence, can uniquely be expressed through each other. Consequently, if the Hamiltonian $\mathcal{H}$ is known, one can construct from it the derivation D and *vice versa*. In the next section we shall see that to the Hamiltonian $\mathcal{H}$, as an observable, in the bundle description corresponds, besides D, a suitable lifting H of paths or (multiple-valued) section along paths, the bundle Hamiltonian.

Now we shall derive a new form of the bundle Schrödinger equation in terms of the derivation $\widetilde{D}$ along paths in $\mathrm{mor}_M(F, \pi, M)$ induced by the derivation D along paths generated by the evolution transport U.[40]

Applying equation (6.20), we can find the explicit (matrix of the) action of $\widetilde{D}^\gamma_t(C) := D^\gamma_t \circ C$, $C \in \mathrm{PLift}^1(\mathrm{mor}_M(F, \pi, M))$, on a state lifting Ψ provided the lift C_γ is *linear*.

[39] This initial-value problem is analogous (and equivalent) to the one for the Schrödinger equation (2.6) and condition (2.7).

[40] For the notation and corresponding definitions, see Subsect. 3.3, in particular, equations (3.33)–(3.37).

Let $\mathbf{[}X\mathbf{]}$ be the matrix of a vector or an operator X in $\{e_a\}$. Due to (6.20), we have

$$\mathbf{[}(\tilde{D}_t^\gamma(C))\Psi\mathbf{]} = \left(\frac{d}{dt}\boldsymbol{C}_\gamma(t)\right)\boldsymbol{\Psi}_\gamma(t) + \boldsymbol{C}_\gamma(t)\left(\frac{d}{dt}\boldsymbol{\Psi}_\gamma(t)\right) + \boldsymbol{\Gamma}_\gamma(t)\boldsymbol{C}_\gamma(t)\boldsymbol{\Psi}_\gamma(t),$$

which is a special case of (3.50). Substituting here $\frac{d}{dt}\boldsymbol{\Psi}_\gamma(t)$ from (6.12) and using (6.22), we obtain the matrix equation

$$\mathbf{[}(\tilde{D}_t^\gamma(C))\Psi\mathbf{]} = \left(\frac{d}{dt}\boldsymbol{C}_\gamma(t)\right)\boldsymbol{\Psi}_\gamma(t) + [\boldsymbol{\Gamma}_\gamma(t), \boldsymbol{C}_\gamma(t)]_{\_}\,\boldsymbol{\Psi}_\gamma(t), \tag{6.32}$$

where $[\cdot,\cdot]_{\_}$ denotes the commutator of matrices, or

$$\mathbf{[}\tilde{D}_t^\gamma(C)\mathbf{]} = \frac{d}{dt}\boldsymbol{C}_t + [\boldsymbol{\Gamma}_\gamma(t), \boldsymbol{C}_\gamma(t)]_{\_}\,. \tag{6.33}$$

Comparing this equation with (3.49), we get $\mathbf{[}\widetilde{D}_t^\gamma(C)\mathbf{]} = \mathbf{[}\,^\circ D_t^\gamma(C)\mathbf{]}$ where $^\circ D$ is the derivation along paths in $\mathrm{mor}_M(F,\pi,M)$ associated to D according to (3.48). Therefore the invariant bundle form of (6.33) is

$$\widetilde{D}(C) = {}^\circ D(C), \tag{6.34}$$

where $C \in \mathrm{PLift}^1(\mathrm{mor}_M(F,\pi,M))$ acts *only on state liftings* according to (3.34) and C_γ is *linear*, or, equivalently, we can write

$$\widetilde{D}|_{\mathcal{O}} = {}^\circ D|_{\mathcal{O}} \tag{6.35}$$

with $\mathcal{O}$ being the set of just-described liftings C.

We derived (6.34) under the assumption that C_γ is linear and C acts on state liftings, i.e. on ones satisfying the matrix-bundle Schrödinger equation (6.12). Conversely, if we apply (6.33) to some vector $\Phi_\gamma(t) \in F_{\gamma(t)}$ and compare the result with the one for $(D_t^\gamma(C))\,(\Phi)$ obtained through (6.20) (see above), we see that $\Phi_\gamma(t)$ satisfies (6.12). Consequently, equation (6.34) with linear C_γ is valid if and only if C is applied on liftings representing the evolution of a quantum system. Hence Ψ is a state lifting, i.e. it satisfies, for instance, the bundle Schrödinger equation (6.27), iff the equation

$$(\widetilde{D}(C))\Psi = ({}^\circ D(C))\Psi, \tag{6.36}$$

is valid for every lifting C in the bundle of restricted morphisms such that C_γ is linear for every γ. In particular (6.36) is valid for the (Hermitian) liftings (of paths) corresponding to observables (see further Sect. 7) and Ψ satisfying the bundle Schrödinger equation (6.27).

The over-all above discussion shows the equivalence of (6.36) (for every C with C_γ linear) with the Schrödinger equation (in anyone of its (equivalent)

forms mentioned until now). That is why (6.36) can be called *matrix-lifting Schrödinger equation.*

We want to point to a substantial difference between, from one hand, the bundle Schrödinger equation (6.27), or (6.36), or (6.30) and, from other hand, the initial conditions for it (see (6.31) or (6.28)) or the conventional Schrödinger equation (2.6) and the initial conditions (2.7) for it. The bundle Schrödinger equations are absolutely invariant in a sense that they do not depend on some coordinates, space(-time) points, or reference paths like γ and hence, in our interpretation, are observer-independent. In this attitude, the bundle Schrödinger equations are analogous to the covariant equations in general relativity which, due to their tensorial character, have similar properties. In contrast to the mentioned observation, the initial bundle conditions depend on the reference path γ, i.e. are observer-dependent as, e.g., the conventional Hamiltonian $\mathcal{H}$ is such.[41] Consequently in the Hilbert bundle description the observer-dependence, i.e. the dependence on the reference path γ, is 'moved' from the equations of motion to the initial conditions for them. It is clear, this dependence cannot be removed completely due to the equivalence between the Hilbert space and Hilbert bundle descriptions of quantum mechanics.

Since now we have in our disposal the machinery required for analysis of [35], we, as promised in Sect. 1, want to make some comments on it. In [35, p. 1455, left column, paragraph 4] is stated "that in the Heisenberg gauge (picture) the Hamiltonian is the null operator". If so, all eigenvalues of the Hamiltonian vanish and, as they are picture-independent, they are null in any picture of quantum mechanics. Consequently, form here one deduces the absurd conclusion that the 'energy levels of any system coincide and correspond to one and the same energy equal to zero'. Since the paper [35] is mathematically completely correct and rigorous, there is something wrong with the physical interpretation of the mathematical scheme developed in it. Without going into details, we describe below the solution of this puzzle which simultaneously throws a bridge between [35] and the present investigation.

In [35] the system's Hilbert space $\mathfrak{H}$ is replace by a differentiable Hilbert bundle $\mathfrak{E}(\mathbb{R}_+, \mathfrak{H})$ (in our terms $(\mathfrak{E}, \pi, \mathbb{R}_+)$ with a fibre $\mathfrak{H}$), $\mathbb{R}_+ := \{t : t \in \mathbb{R}, t \geq 0\}$, which is an associated Hilbert bundle of the principle fibre bundle $\mathfrak{P}\big(\mathbb{R}_+, \mathfrak{U}(\mathfrak{H})\big)$ of orthonormal bases of $\mathfrak{H}$ where $\mathfrak{U}(\mathfrak{H})$ is the unitary group of (linear) bounded invertible operators in $\mathfrak{H}$ with bounded inverse. Let $p : \mathfrak{U}(\mathfrak{H}) \to GL(\mathbb{C}, \dim \mathfrak{H})$ be a (linear and continuous) representation of

[41] For instance, suppose two point-like free particles 1 and 2 have masses m_a and momentum operators p_a, $a = 1, 2$ with respect to some observer. The particle's Hamiltonians are $\mathcal{H}_a = p_a^2/2m$, $a = 1, 2$. The Hamiltonian of the second particle with respect to the first one is $\mathcal{H}_{1,2} = p^2/2m$ (after the elimination of the center of mass movement) with $p := (m_2 p_1 - m_1 p_2)/m_1 m_2$ and $m := m_1 m_2/(m_1 + m_2)$. For details, see [63, chapter IX, §§11, 12].

$\mathfrak{U}(\mathfrak{H})$ into the general linear group of $\dim \mathfrak{H}$-dimensional matrices. An obvious observation is that [35, equation (4.6)] under p transforms, up to notation, to our equation (6.24) (in [35] is taken $\hbar = 1$). Thus we see that what in [35] is called Hamiltonian is actually the (analogue of the) matrix-bundle Hamiltonian $\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t)$, not the Hamiltonian $\mathcal{H}$ itself. This immediately removes the above-pointed conflict: as we shall see later (see Sect. 8, equation (8.5)) along any γ (or, over $\mathbb{R}_+$ in the notation of [35] - see below), we can choose a field of frames (bases) in which $\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t)$ identically vanishes but, due to (6.13), this does not imply the vanishment of the Hamiltonian at all. This particular choice of the frame along γ corresponds to the 'Heisenberg gauge' in [35], normally known as Heisenberg picture.

Having in mind the above, we can describe [35] as follows. In it we have $F = \mathfrak{E}$, $M = \mathbb{R}_+$, $\mathcal{F} = \mathfrak{H}$ (the conventional system's Hilbert space), $J = \mathbb{R}_+$, $\gamma = \mathsf{id}_{\mathbb{R}_+}$ (other choices of γ correspond to reparametrization of the time), and $\frac{\partial}{\partial t}$, $t \in \mathbb{R}_+$ is the analog of D^+ in [35]. As we already pointed, the matrix-bundle Hamiltonian $\boldsymbol{H}^{\mathbf{m}}_{\gamma}(t)$ represents the operator $A(t)$ of [35], incorrectly identified there with the 'Hamiltonian' and the choice of a field of bases over $\gamma(J) = \mathbb{R}_+ = M$ corresponds to an appropriate 'choice of the gauge' in [35]. Now, after its correspondence between [35] and the present work is set, one can see that under the representation p the main results of [35], expressed by [35, equations (4.5), (4.6) and (4.8], correspond to our equations (6.25) (see also (6.20)), (6.24) and (6.5) respectively.

Ending with the comment on [35], we note two things. First, this paper uses a rigorous mathematical base, analogous to the one in [10], which is not a goal of our work. And, second, the ideas of [35] are a very good motivation for the present investigation and are helpful for its better understanding.

7. The bundle description of observables

In quantum mechanics is accepted that to any dynamical variable, say $\mathbb{A}$, there corresponds a unique observable, say $\mathcal{A}(t)$, which is a Hermitian linear operator in the Hilbert space $\mathcal{F}$, i.e. $\mathcal{A}(t)\colon \mathcal{F} \to \mathcal{F}$, $\mathcal{A}(t)$ is linear, and $\mathcal{A}^{\dagger} = \mathcal{A}$ [8–10].

The mean value of an observable $\mathcal{A}$ in a state with state vector $\psi \in \mathcal{F}$ with finite norm is calculated according to (2.11). It is interpreted as an observed (mean) value of the dynamical variable $\mathbb{A}$ at a state ψ. This assumption and the probabilistic interpretation of the wave function ψ are the main tools for predicting experimentally observable results in quantum mechanics. As we said earlier in Subsect 4.3, the latter of these tools is transferred in Hilbert bundle quantum mechanics in an evident way. The bundle version of the former one is the main task of this section. Below will be shown that the proper bundle analogue of $\mathcal{A}$ is a suitable lifting of paths (in the bundle of restricted morphisms of the Hilbert bundle of states) or a

(generally multiple-valued) morphism along paths (in the system's Hilbert bundle).

7.1. Heuristic introduction

Let $\psi^{(\lambda)}(t) \in \mathcal{F}$ be an eigenvector of $\mathcal{A}(t)$ with eigenvalue λ ($\in \mathbb{R}$), i.e. $\mathcal{A}(t)\psi^{(\lambda)}(t) = \lambda\psi^{(\lambda)}(t)$. According to (4.3) to $\psi^{(\lambda)}(t)$ corresponds the vector $\Psi^{(\lambda)}_{\gamma}(t) = l^{-1}_{\gamma(t)}\psi^{(\lambda)}(t) \in F_{\gamma(t)}$ in the bundle description. But the Hilbert space and Hilbert bundle descriptions of a quantum evolution should be fully equivalent. Hence to $\mathcal{A}(t)$ in $F_{\gamma(t)}$ should correspond certain operator which we denote by $A_{\gamma}(t)$. We define this operator by demanding every $\Psi^{(\lambda)}_{\gamma}(t)$ to be its eigenvector with eigenvalue λ, i.e. $\big(A_{\gamma}(t)\big)\,\Psi^{(\lambda)}_{\gamma}(t) := \lambda\Psi^{(\lambda)}_{\gamma}(t)$. Combining this equality with the preceding two, we easily verify that $\big(A_{\gamma}(t)\circ l^{-1}_{\gamma(t)}\big)\psi^{(\lambda)}(t) = \big(l^{-1}_{\gamma(t)}\circ\mathcal{A}(t)\big)\psi^{(\lambda)}(t)$ where the linearity of l_x has been used. Admitting that $\{\psi^{(\lambda)}(t)\}$ is a complete set of vectors, i.e. a basis of $\mathcal{F}$, we find

$$A_{\gamma}(t) = l^{-1}_{\gamma(t)}\circ\mathcal{A}(t)\circ l_{\gamma(t)}\colon F_{\gamma(t)} \to F_{\gamma(t)}. \tag{7.1}$$

More 'physically', the same result is derivable from (2.11) too. The mean value $\langle\mathcal{A}\rangle^t_{\psi}$ of $\mathcal{A}$ at a state $\psi(t)$ is given by (2.11) and the mean value of $A_{\gamma}(t)$ at a state $\Psi_{\gamma}(t)$ is

$$\langle A_{\gamma}(t)\rangle_{\Psi_{\gamma}(t)} := \langle A_{\gamma}(t)\rangle^t_{\Psi_{\gamma}} := \frac{\langle\Psi_{\gamma}(t)|A_{\gamma}(t)\Psi_{\gamma}(t)\rangle_{\gamma(t)}}{\langle\Psi_{\gamma}(t)|\Psi_{\gamma}(t)\rangle_{\gamma(t)}}, \tag{7.2}$$

i.e. it is given via (2.11) in which the scalar product $\langle\cdot|\cdot\rangle_x$, defined by (3.1), is used instead of $\langle\cdot|\cdot\rangle$. We define $A_{\gamma}(t)$ by demanding

$$\langle\mathcal{A}(t)\rangle^t_{\psi} = \langle A_{\gamma}(t)\rangle^t_{\Psi_{\gamma}}. \tag{7.3}$$

Physically this condition is quite natural as it means that the observed values of the dynamical variables are independent of the way we calculate them. From this equality, (4.1), and (3.1), we get $\langle\psi(t)|\mathcal{A}(t)\psi(t)\rangle = \langle\psi(t)|l_{\gamma(t)}\circ A_{\gamma}(t)\circ l^{-1}_{\gamma(t)}\psi(t)\rangle$ which, again, leads to (7.1). Thus we have also proved the equivalence of (7.1) and (7.3).

The above considerations lead to the idea that to every observable $\mathcal{A}$ at a moment t there should correspond an operator $A_{\gamma}(t)$, given by (7.1), in the fibre $F_{\gamma(t)} = \pi^{-1}(\gamma(t))$. It is almost evident, if $\gamma\colon J \to M$ is without self-intersections, the collection of maps $\{A_{\gamma}(t)|t \in J\}$ forms a morphism over $\gamma(J)$ of the restricted on $\gamma(J)$ system's Hilbert bundle.

7.2. Rigorous considerations

As it was mentioned earlier (see Subsect. 4), postulates 4.1 and 4.2 are not enough for the bundle description of observables. The contents of Subsect. 7.1 confirms this opinion. Relying on the above not quite rigorous results, we formulate the missing section of the chain as the next postulate.

Postulate 7.1. *Let (F, π, M) be the Hilbert bundle of a quantum system, $\gamma\colon J \to M$, and $t \in J$. In the bundle description of quantum mechanics, every dynamical variable $\mathbb{A}$ characterizing the system is represented by a map A assigning to the pair (γ, t) a map $A_\gamma(t)\colon \pi^{-1}(\gamma(t)) \to \pi^{-1}(\gamma(t))$ such that*

$$A_\gamma(t) = l^{-1}_{\gamma(t)} \circ \mathcal{A}(t) \circ l_{\gamma(t)} \tag{7.4}$$

where $\mathcal{A}(t)\colon \mathcal{F} \to \mathcal{F}$ is the linear Hermitian operator (in the system's Hilbert space $\mathcal{F}$) representing $\mathbb{A}$ in the conventional quantum mechanics. If at a moment $t \in J$ the system is in a state characterized by a bundle state vector $\Psi_\gamma(t)$ with a finite norm (in $F_{\gamma(t)}$), the observed value of $\mathbb{A}$ (or of A) with respect to γ at a moment t is equal to the mean value of $A_\gamma(t)$ in $\Psi_\gamma(t)$ which, by definition, is given by (7.2).

From (7.4), (7.2), (4.3), and (2.11), we derive (7.3). This simple result has a fundamental meaning: the observed values of a dynamical variable are (and must be!) independent of the way they are calculated. This assertion may be called '*principle of invariance of the observed (mean) values*' and its essence is the independence of the physically measurable quantities of the mathematical way we describe them. In our context, it means the coincidence of the observed values of a dynamical variable calculated in the Hilbert bundle and Hilbert space descriptions. In other words, we can express the same by saying that the predictions of both, conventional and bundle, versions of quantum mechanics are absolutely identical regardless of the existence of three free parameters (the base M, the set $\{l_x | x \in M\}$, and the path γ) in the bundle case.

Let us now clarify the mathematical nature of the mapping A introduced via postulate 7.1. First of all, the maps $A_\gamma(t)$ are linear as $\mathcal{A}$ and $l_{\gamma(t)}$ are such (see (7.4)). If we define A as a map $A\colon \gamma \mapsto A_\gamma$ with $A_\gamma\colon t \mapsto A_\gamma(t)$, we see that $A_\gamma\colon J \to F_0^M$, where

$$\begin{aligned} F_0^M &:= \{\varphi_x \,|\, \varphi_x\colon F_x \to F_x,\ x \in M\} \\ &= \{\varphi_x \,|\, \varphi_x = \varphi|_{F_x},\ x \in M,\ \varphi \in \mathrm{Mor}_B(F, \pi, M)\} \end{aligned}$$

is the bundle space of the bundle $\mathrm{mor}_M(F, \pi, M)$ of restricted morphisms over M (see Subsect. 4.1). Since the bundles (F, π, M) and $\mathrm{mor}_M(F, \pi, M)$ have a common base, the manifold M, we conclude that A_γ is a lifting

of $\gamma\colon J \to M$ in $\mathrm{mor}_M(F,\pi,M)$ (not in (F,π,M)!). Consequently, the map A, as considered above, is a *lifting of paths in the bundle of restricted M-morphisms of the system's Hilbert bundle of states*,

$$A \in \mathrm{PLift}\big(\mathrm{mor}_M(F,\pi,M)\big). \tag{7.5}$$

The linear maps $A_\gamma(t)\colon F_{\gamma(t)} \to F_{\gamma(t)}$ are Hermitian. Indeed, using (3.7) and (3.8) for $y = x = \gamma(t)$ and $A_{x\to x} = A_\gamma(t)$, and (7.4), we get

$$A^\ddagger_\gamma(t) = A_\gamma \tag{7.6}$$

where the Hermiticity of $\mathcal{A}$ was used. A lift A_γ in $\mathrm{mor}_M((F,\pi,M))$ of $\gamma\colon J \to M$ is called Hermitian if (7.6) holds for every $t \in J$; we denote this by writing symbolically $A^\ddagger_\gamma = A_\gamma$. Respectively, a lifting A in $\mathrm{PLift}(\mathrm{mor}_M(F,\pi,M))$ is Hermitian, $A^\ddagger = A$, if $A\colon \gamma \mapsto A_\gamma$ and $A^\ddagger_\gamma = A_\gamma$ for every path $\gamma \in \mathrm{P}(M)$ in M.

Let us summarize. In the bundle description a dynamical variable $\mathbb{A}$ is represented by a Hermitian lifting A of paths in the bundle of restricted morphisms over the base in the Hilbert bundle of states. For A equations (7.4) holds and its mean value along γ at a moment t for a system with state lifting Ψ is

$$\langle A\rangle^{t,\gamma}_\Psi := \langle A_\gamma(t)\rangle^t_{\Psi_\gamma} \tag{7.7}$$

with the r.h.s. of this equality given by (7.2).

The map A, provided via postulate 7.1, can also be considered as a (multiple-valued) morphism along paths of the Hilbert bundle of states.[42] From one hand, define $A\colon \gamma \mapsto {}_\gamma A$ with ${}_\gamma A\colon x \mapsto \{A_\gamma(t)|\gamma(t) = x,\ t \in J\}$ for $x \in \gamma(J)$. If γ is without self-intersections, then ${}_\gamma A$ is in $\mathrm{Mor}_{\gamma(J)}(F,\pi,M)|_{\gamma(J)}$ (see Subsect. 4.1). From other hand, we can defined $A\colon \gamma \mapsto {}_\gamma A$ as a map ${}_\gamma A\colon \pi^{-1}(\gamma(J)) \to \pi^{-1}(\gamma(J))$ with ${}_\gamma A|_{\pi^{-1}(x)} = \{A_\gamma(t)|\gamma(t) = x,\ t \in J\}$. In this case, if γ is without self-intersections, ${}_\gamma A \in \mathrm{Mor}_{\gamma(J)}(F,\pi,M)|_{\gamma(J)}$, i.e. up to a bijective map ${}_\gamma A$ is in $\mathrm{Sec}\big(\mathrm{mor}_{\gamma(J)}(F,\pi,M)|_{\gamma(J)}\big)$. Recalling that a morphism φ over M along paths of a bundle (E,π,B) is a map $\varphi\colon \gamma \mapsto \varphi_\gamma \in \mathrm{Mor}_{\gamma(J)}(F,\pi,M)|_{\gamma(J)}$ for every path $\gamma \in \mathrm{P}(B)$, we see that A is a morphism over M along paths without self-intersections. But if γ is not injective, the map $A\colon \gamma \mapsto {}_\gamma A$ is, generally, multiple-valued morphism (over M) along paths of (F,π,M) and it gives an alternative description of the map A introduced via postulate 7.1. If the multiplicity of A as a morphism along paths is really presented, this description will rarely be employed; if A as a morphism is single-valued, it is somewhat 'simpler' to consider A as a morphism than as a lifting of paths and, respectively, this interpretation will be preferred.

[42] Cf. the analogous situation concerning state liftings and sections in Subsect. 4.3.

Definition 7.1. The unique Hermitian lifting of paths in the bundle of restricted morphisms (over the base of the Hilbert bundle of states) corresponding to a dynamical variable will be called observable lifting (of paths) Respectively, the corresponding (multiple-valued) morphism (over the base) along paths of the Hilbert bundle of states will be called observable morphism (along paths).

By virtue of (7.6), the observable morphisms along paths are Hermitian,

$$A^{\ddagger} = A, \tag{7.8}$$

which is also a corollary of (7.4) and (3.15).

Generally, to every operator $\mathcal{A}\colon \mathcal{F} \to \mathcal{F}$ there corresponds a unique (global) morphism $\overline{A} \in \mathrm{Mor}(F, \pi, M)$ given by

$$\overline{A}_x = \overline{A}\big|_{F_x} = l_x^{-1} \circ \mathcal{A} \circ l_x, \qquad x \in M, \quad \mathcal{A}\colon \mathcal{F} \to \mathcal{F}. \tag{7.9}$$

Consequently to an observable $\mathcal{A}(t)$ can be assigned the (global) morphism $\overline{A}(t)$, $\overline{A}(t)|_{F_x} = l_x^{-1} \circ \mathcal{A}(t) \circ l_x$. But this morphism $\overline{A}(t)$ is almost useless for our goals as it simply gives in any fibre F_x a linearly isomorphic image of the initial observable $\mathcal{A}(t)$ (see Sect. 4).

Notice that $A_\gamma(t)$ generally depends explicitly on t even if $\mathcal{A}$ does not. In fact, from (7.1) we get

$$\frac{\partial \boldsymbol{A}_\gamma(t)}{\partial t} = \big[\boldsymbol{g}_\gamma(t), \boldsymbol{A}_\gamma(t)\big]_{-} + \boldsymbol{l}^{-1}_{\gamma(t)}(t) \frac{\partial \boldsymbol{\mathcal{A}}(t)}{\partial t} \boldsymbol{l}_{\gamma(t)}(t), \tag{7.10}$$

where $[\cdot,\cdot]_{-}$ denotes the commutator of corresponding quantities, and

$$\boldsymbol{g}_\gamma(t) := -\boldsymbol{l}^{-1}_{\gamma(t)}(t) \frac{d\boldsymbol{l}_{\gamma(t)}(t)}{dt}. \tag{7.11}$$

In particular, to the Hamiltonian $\mathcal{H}$ in $\mathcal{F}$ there corresponds the *bundle Hamiltonian* H given by

$$H_\gamma(t) := l^{-1}_{\gamma(t)} \circ \mathcal{H}(t) \circ l_{\gamma(t)}. \tag{7.12}$$

It is an observable lifting of paths or morphism along paths.

From (7.12), using (2.9) and (5.10), we find

$$H_\gamma(t) = \mathrm{i}\hbar l^{-1}_{\gamma(t)} \circ \frac{\partial \mathcal{U}(t, t_0)}{\partial t} \circ l_{\gamma(t_0)} \circ U_\gamma(t_0, t). \tag{7.13}$$

From here a relationship between the matrix-bundle Hamiltonian and the bundle Hamiltonian can be obtained. For this purpose, we write (7.13) in a matrix form and using (6.18) and $df_a(t)/dt = E_a{}^b f_b(t)$, we get:

$$\boldsymbol{H}_\gamma(t) = \boldsymbol{H}^{\mathbf{m}}_\gamma(t) + \mathrm{i}\hbar \boldsymbol{l}^{-1}_{\gamma(t)}(t) \left(\frac{d\boldsymbol{l}_{\gamma(t)}(t)}{dt} + \boldsymbol{E}(t) \boldsymbol{l}_{\gamma(t)}(t) \right). \tag{7.14}$$

Substituting here (6.13), we obtain

$$\boldsymbol{H}_\gamma(t) = \boldsymbol{l}^{-1}_{\gamma(t)}(t)\boldsymbol{\mathcal{H}}(t)\boldsymbol{l}_{\gamma(t)}(t) \tag{7.15}$$

which is simply the matrix form of (7.12). Combining (7.14) with (6.14), we find the following connection between the matrix of the bundle Hamiltonian and the matrix Hamiltonian:

$$\boldsymbol{H}_\gamma(t) = \boldsymbol{l}^{-1}_{\gamma(t)}(t)\boldsymbol{\mathcal{H}}^{\mathbf{m}}(t)\boldsymbol{l}_{\gamma(t)}(t) + \mathrm{i}\hbar\boldsymbol{l}^{-1}_{\gamma(t)}(t)\boldsymbol{E}(t)\boldsymbol{l}_{\gamma(t)}(t). \tag{7.16}$$

Notice, due to (7.9) as well as to (7.1), to the identity map of $\mathcal{F}$ there corresponds a morphism along paths equal to the identity map of F:

$$\mathsf{id}_{\mathcal{F}} \longleftrightarrow \mathsf{id}_F. \tag{7.17}$$

7.3. Functions of observables

The results expressed by (7.1) and (7.9) enable functions of observables in $\mathcal{F}$ to be transferred into ones of liftings of paths (morphisms along paths) or morphisms of (F, π, M), respectively. We will illustrate this in the case of, e.g., two variables.

Let $\mathcal{G}\colon (\mathcal{A}(t), \mathcal{B}(t)) \mapsto \mathcal{G}(\mathcal{A}(t), \mathcal{B}(t))\colon \mathcal{F} \to \mathcal{F}$ be a function of the observables $\mathcal{A}(t), \mathcal{B}(t)\colon \mathcal{F} \to \mathcal{F}$. It is natural to define the bundle analogue G of $\mathcal{G}$ by

$$G\colon (A, B) \mapsto G(A, B)\colon \gamma \mapsto G_\gamma(A, B)\colon \pi^{-1}(\gamma(J)) \to \pi^{-1}(\gamma(J)),$$

where $G_\gamma(A, B)$ is a lifting of γ and

$$\begin{aligned} G_\gamma(A, B)|_t &:= l^{-1}_{\gamma(t)} \circ \mathcal{G}(\mathcal{A}(t), \mathcal{B}(t)) \circ l_{\gamma(t)} \\ &= l^{-1}_{\gamma(t)} \circ \mathcal{G}(l_{\gamma(t)} \circ A_\gamma(t) \circ l^{-1}_{\gamma(t)}, l_{\gamma(t)} \circ B_\gamma(t) \circ l^{-1}_{\gamma(t)}) \circ l_{\gamma(t)}. \end{aligned} \tag{7.18}$$

Thus $G(A, B)$ is an observable lifting of paths. This definition becomes evident in the cases when $\mathcal{G}$ is a polynom or if it is expressible as a convergent power series; in both cases the multiplication has to be understood as an operator composition. If we are dealing with one of these cases, the definition (7.18) follows from the fact that for any observable liftings $A_1, \ldots, A_k$, $k \in \mathbb{N}$ of paths, the equality

$$A_{1,\gamma}(t) \circ A_{2,\gamma}(t) \circ \cdots \circ A_{k,\gamma}(t) = l^{-1}_{\gamma(t)} \circ (\mathcal{A}_1(t) \circ \mathcal{A}_2(t) \circ \cdots \circ \mathcal{A}_k(t)) \circ l_{\gamma(t)} \tag{7.19}$$

holds due to (7.1). In these cases $G(A, B)$ depends only on A and B and it is explicitly independent on the isomorphisms l_x, $x \in M$.

The commutator of two operators is a an important operator function in quantum mechanics. In the Hilbert space and bundle descriptions it is

defined, respectively, by $[\mathcal{A},\mathcal{B}]_- := \mathcal{A}\circ\mathcal{B}-\mathcal{B}\circ\mathcal{A}$ and $[A,B]_- := A\circ B - B\circ A$, where (see (7.18)) $(A\circ B)\colon \gamma\mapsto (A\circ B)_\gamma t\colon \mapsto (A\circ B)_\gamma(t) = A_\gamma(t)\circ B_\gamma(t)$. The relation

$$[A_\gamma(t), B_\gamma(t)]_- = l^{-1}_{\gamma(t)}\circ[\mathcal{A},\mathcal{B}]_-\circ l_{\gamma(t)} \tag{7.20}$$

is an almost evident corollary of (7.1). It can also be considered as a special case of (7.18). In particular, to commuting observables (in $\mathcal{F}$) there correspond commuting observable liftings or morphisms:

$$[\mathcal{A},\mathcal{B}]_- = 0 \iff [A,B]_- = 0. \tag{7.21}$$

A little more general is the result, following from (7.20), that to observables whose commutator is a c-number there correspond observable liftings with the same c-number as a commutator:

$$[\mathcal{A},\mathcal{B}]_- = c\,\mathsf{id}_{\mathcal{F}} \iff [A,B]_- = c\,\mathsf{id}_F. \tag{7.22}$$

for some $c\in\mathbb{C}$. In particular, the bundle analogue of the famous relation $[\mathcal{Q},\mathcal{P}]_- = \mathrm{i}\hbar\,\mathsf{id}_{\mathcal{F}}$ between a coordinate $\mathcal{Q}$ and the conjugated to it momentum $\mathcal{P}$ is $[Q,P]_- = \mathrm{i}\hbar\,\mathsf{id}_F$.

A bit more complicated is the case for operators and liftings of paths at different 'moments'. Let $\gamma\colon J\to M$ and $r,s,t\in J$. If $\breve{\mathcal{G}}_{s,t}\colon(\mathcal{A},\mathcal{B})\mapsto \mathcal{G}(\mathcal{A}(s),\mathcal{B}(t))$, we define the bundle analogue $\breve{G}_{s,t}$ of $\breve{\mathcal{G}}_{s,t}$ by

$$\breve{G}_{s,t}\colon (A,B)\mapsto \breve{G}_{s,t}(A,B)\colon \gamma\mapsto \breve{G}_{\gamma;s,t}(A,B)\colon \pi^{-1}(\gamma(J))\to\pi^{-1}(\gamma(J)),$$

where

$$\begin{aligned}&\breve{G}_{\gamma;s,t}(A,B)\Big|_r := l^{-1}_{\gamma(r)}\circ\mathcal{G}(\mathcal{A}(s),\mathcal{B}(t))\circ l_{\gamma(r)}\\ &= l^{-1}_{\gamma(r)}\circ\mathcal{G}(l_{\gamma(r)}\circ\breve{\mathcal{A}}_{\gamma;s}(r)\circ l^{-1}_{\gamma(r)}, l_{\gamma(r)}\circ\breve{\mathcal{B}}_{\gamma;t}(r)\circ l^{-1}_{\gamma(r)})\circ l_{\gamma(r)}\colon F_{\gamma(r)}\to F_{\gamma(r)}.\end{aligned} \tag{7.23}$$

Here

$$\breve{\mathcal{A}}_{\gamma;t}(r) := l^{-1}_{\gamma(r)}\circ\mathcal{A}(t)\circ l_{\gamma(r)} = l^{\gamma}_{t\to r}\circ A(t)\circ l^{\gamma}_{r\to t}\colon F_{\gamma(r)}\to F_{\gamma(r)}, \tag{7.24}$$

where (7.1) has been used and $l^{\gamma}_{s\to t} := l_{\gamma(s)\to\gamma(t)}$ is the (flat) linear transport (along paths) from $\gamma(s)$ to $\gamma(t)$ assigned to the isomorphisms l_x, $x\in M$ (see equation (3.13)).[43] Now the analogue of (7.19) is

$$\begin{aligned}&\breve{\mathcal{A}}_{1;\gamma;t_1}(r)\circ\breve{\mathcal{A}}_{2;\gamma;t_2}(r)\circ\cdots\circ\breve{\mathcal{A}}_{k;\gamma;t_k}(r)\\ &\qquad = l^{-1}_{\gamma(r)}\circ(\mathcal{A}_1(t_1)\circ\mathcal{A}_2(t_2)\circ\cdots\circ\mathcal{A}_k(t_k))\circ l_{\gamma(r)}.\end{aligned} \tag{7.25}$$

[43] According to [59, sections 2 and 3] the observable lifting $\breve{A}_{\gamma;t}(r)$ along γ is obtained via linear transportation of $A_\gamma(t)$ along γ by means of the induced by $l^{\gamma}_{s\to t}$ linear transport along paths in the bundle $\mathrm{mor}_M(F,\pi,M)$ of restricted morphisms over M of (F,π,M) (see also subsection 8.1).

So, if $\mathcal{G}$ is a polynom or a convergent power series, the observable lifting $\breve{G}_{\gamma;s,t}(A,B)$ along γ depends only on $\breve{A}_{\gamma;s}(r)$ and $\breve{B}_{\gamma;t}(r)$.

In particular for $\mathcal{G}(\cdot,\cdot)=[\cdot,\cdot]_-$, have

$$\left[\breve{A}_{\gamma;s}(r),\breve{B}_{\gamma;t}(r)\right]_- = l^{-1}_{\gamma(r)}\circ[\mathcal{A}(s),\mathcal{B}(t)]_-\circ l_{\gamma(r)} \tag{7.26}$$

which for $s=r=t$ reduces to (7.20). In this case the analogues of (7.21) and (7.22) are

$$[\mathcal{A}(s),\mathcal{B}(t)]_-=0 \iff \left[\breve{A}_{\gamma;s}(r),\breve{B}_{\gamma;t}(r)\right]_-=0, \tag{7.27}$$

$$[\mathcal{A}(s),\mathcal{B}(t)]_-=c\,\mathsf{id}_{\mathcal{F}} \iff \left[\breve{A}_{\gamma;s}(r),\breve{B}_{\gamma;t}(r)\right]_-=c\,\mathsf{id}_{F_{\gamma(r)}}, \tag{7.28}$$

respectively.

The above considerations can *mutatis mutandis*, e.g. by replacing $\gamma(t)$ with x, $\mathcal{A}(t)$ with $\mathcal{A}$, A with $\overline{A}$, etc., be transferred to global morphisms of (F,π,M), but this is not required for the present investigation.

8. Pictures of motion from bundle view-point

Well-known are the different pictures (or representations) of motion of a quantum system [9, ch. VIII, §§ 9, 10, 14], [8, ch. III, § 14], [7, § 27, § 28]: the Schrödinger's, Heisenberg's, interaction's, and other 'intermediate' ones. Nevertheless that they are equivalent from the view-point of physically predictable results, these special representations of the quantum-mechanical formalism reflect its different sides. Correspondingly, the choice of a concrete picture depends on the particular physical problem under investigation. Bellow we consider certain general problems connected with these special pictures of motion of a quantum system from the fibre bundle view-point on quantum mechanics proposed in this investigation.

8.1. Schrödinger picture

In fact, the bundle Schrödinger picture of motion of a quantum system is the way of its description we have been dealing until now [2, 3]. Its basic assertions will be summarize in this subsection.

The states of a quantum system form a Hilbert bundle (F,π,M) whose base M is a C^1 manifold, interpreted as a space(-time) model. The system state is described by a lifting Ψ of paths over (F,π,M),

$$\Psi\in\mathrm{PLift}(F,\pi,M), \tag{8.1}$$

which also admits equivalent interpretation as a, generally, multiple-valued section along paths in the same bundle. Along every path $\gamma\colon J\to M$, interpreted as a trajectory (world line) of some observer, the different time-values

of the bundle state vectors $\Psi_\gamma(t)$ are connected via equation (5.7) in which $U_\gamma(t,s)$ is the evolution transport along γ from s to t, $s,t \in J$. The state lifting Ψ is generically a variable in time quantity evolving according to the bundle Schrödinger equation (6.27). This equation, together with some initial condition, is equivalent to the Schrödinger equation (initial-value problem) (6.28) for the evolution transport U.

In the bundle description to a dynamical variable $\mathbb{A}$ corresponds a unique lifting of paths A in the bundle of restricted morphisms of the system's Hilbert bundle of states,

$$A \in \mathrm{PLift}(\mathrm{mor}_M(F,\pi,M)) \tag{8.2}$$

The observable lifting A admits also a treatment as a, generally, multiple-value section along paths of (F,π,M). With respect to a reference path $\gamma\colon J \to M$ at some moment $t \in J$ an observable lifting A reduces to a map $A_\gamma(t)\colon F_{\gamma(t)} \to F_{\gamma(t)}$ which, generally, is time-dependent regardless that in the Hilbert space description the corresponding observable $\mathcal{A}$ may happen to be time-independent. It (or its evolution) is explicitly given by (7.1).

Geometrically the maps $A_\gamma(t)$ 'live' in the bundle space

$$\begin{aligned} F_0^M &:= \{\varphi_x \,|\, \varphi_x\colon F_x \to F_x,\ x \in M\} \\ &= \{\varphi_x \,|\, \varphi_x = \varphi|_{F_x},\ x \in M,\ \varphi \in \mathrm{Mor}_B(F,\pi,M)\} \end{aligned} \tag{8.3a}$$

of the bundle of point-restricted morphisms of (F,π,M) whose projection is

$$\pi_0^M\colon F_0^M \to M, \quad \pi_0^M(\varphi) = x_\varphi \tag{8.3b}$$

for $\varphi \in F_0^M$, where $x_\varphi \in M$ is the unique element of M such that $\varphi\colon F_{x_\varphi} \to F_{x_\varphi}$.

Since equations (4.5) and (7.3) are valid, the probabilistic interpretation of conventional quantum mechanics is retained and the predictions of Hilbert bundle and Hilbert space versions of quantum mechanics are identical.

Summing up, in the bundle Schrödinger picture, both the state liftings and observable liftings generically change in time in the corresponding bundles as described above.

8.2. Heisenberg picture

The Heisenberg picture is suitable for analyzing some quantum properties of the systems, as well as for the comparison between classical and quantum mechanics. In it the time-dependence is entirely shift to the dynamical variables, i.e. to the observables representing them, while the state vectors remain constant in time. In this subsection will be proved that analogous transformation is available in the bundle version of quantum mechanics too.

Below we present two different ways for introduction of bundle Heisenberg picture leading, of course, to one and the same result. The first one is based entirely on the bundle approach and reveals its natural geometric character. The second one is a direct analogue of the usual way in which one arrives to this picture.

8.2.1. Hilbert bundle introduction

According to [31, sect. 4] or [32, sect. 3] *every linear transport along paths is locally Euclidean*, i.e. (see [31, sect. 4] for details and rigorous results) along every path there is a field of (generally multiple-valued [31, remark 4.2]) bases, called normal, in which its matrix is unit. Such a collection of bases is called a *normal frame* along the corresponding path. In particular, along $\gamma\colon J \to M$ there exists a frame $\big\{\{\tilde{e}^{\gamma}_a(t)\} - \text{basis in } F_{\gamma(t)}\big\}$ in which the matrix of the evolution transport $U_\gamma(t,s)$ is $\widetilde{\boldsymbol{U}}_\gamma(t,s) = \mathbb{1}$. Explicitly we can put

$$\tilde{e}^{\gamma}_a(t) = U_\gamma(t,t_0)e^{\gamma}_a(t_0), \tag{8.4}$$

where $t, t_0 \in J$, γ is not a summation index, and the basis $\{e^{\gamma}_a(t_0)\}$ in $F_{\gamma(t_0)}$ is fixed [32, proof of proposition 3.1] (cf. [31, equation (4.2)]).[44] Because of (5.9), (6.21), and (6.22) this class of frames normal along γ for the evolution transport is uniquely defined by any one of the (equivalent) equalities:

$$\widetilde{\boldsymbol{U}}_\gamma(t,t_0) = \mathbb{1}, \quad \widetilde{\boldsymbol{\Gamma}}_\gamma(t) = \boldsymbol{0}, \quad \widetilde{\boldsymbol{H}^{\mathfrak{m}}_\gamma}(t) = \boldsymbol{0}. \tag{8.5}$$

So, the matrix-bundle Hamiltonian vanishes in such a special frame and, consequently (see (6.12)), the components of the bundle state vectors remain constant in time t, i.e. $\widetilde{\boldsymbol{\Psi}}_\gamma(t) = \text{const}$, but the vectors themselves are not necessary such as the normal frames along γ are generally time-dependent.

In the normal frame $\{\tilde{e}^{\gamma}_a(t)\}$, defined above by (8.4), the components of $A_\gamma(t)$ are

$$\begin{aligned}
\widetilde{(A_\gamma(t))}_{ab} &= \big\langle \tilde{e}^{\gamma}_a(t) \big| \big(A_\gamma|_{F_{\gamma(t)}} \big) \tilde{e}^{\gamma}_b(t) \big\rangle_{\gamma(t)} \\
&= \langle U_\gamma(t,t_0)e^{\gamma}_a(t_0) | A_\gamma(t)\, U_\gamma(t,t_0)e^{\gamma}_b(t_0)\rangle_{\gamma(t)} \\
&= \langle e^{\gamma}_a(t_0) | \, U^{-1}_\gamma(t,t_0) A_\gamma(t)\, U_\gamma(t,t_0)e^{\gamma}_b(t_0)\rangle_{\gamma(t)} = \big(A^{H}_{\gamma,t}(t_0)\big)_{ab},
\end{aligned}$$

where

$$A^{\mathrm{H}}_{\gamma,t}(t_0) := U^{-1}_\gamma(t,t_0) \circ A_\gamma(t) \circ U_\gamma(t,t_0)\colon F_{\gamma(t_0)} \to F_{\gamma(t_0)}. \tag{8.6}$$

[44] The so-defined field of bases are not uniquely defined at the points of self-intersection, if any, of γ. Evidently, they are unique on any 'part' of γ without self-intersections. The last case covers the interpretation of γ as an observer's world line, in which it cannot have self-intersections. See [31, sect. 4] for details.

Hence the matrix elements of $A_\gamma(t)$ in $\{\tilde{e}_a^\gamma(t)\}$ coincide with those of $A^{\mathrm{H}}_{\gamma,t}(t_0)$ in $\{e_a^\gamma(t_0)\}$. Consequently, due to (2.11), (7.2), (7.3), and (5.14), the mean value of A (along γ) is

$$\begin{aligned}\langle A_\gamma(t)\rangle^t_{\Psi_\gamma} &= \Big(\widetilde{A_\gamma(t)}\Big)_{ab}\,\widetilde{\Psi}^a_\gamma(t)\widetilde{\Psi}^b_\gamma(t)/\langle\Psi_\gamma(t)|\Psi_\gamma(t)\rangle_{\gamma(t)}\\ &= \Big(A^{\mathrm{H}(t_0)}_{\gamma,t}\Big)_{ab}\,\widetilde{\Psi}^a_\gamma(t)\widetilde{\Psi}^b_\gamma(t)/\langle\Psi_\gamma(t_0)|\Psi_\gamma(t_0)\rangle_{\gamma(t)}.\end{aligned}$$

But $\widetilde{\boldsymbol{\Psi}}_\gamma(t) = \widetilde{\boldsymbol{\Psi}}_\gamma(t_0)$, hereout

$$\langle A_\gamma(t)\rangle^t_{\Psi_\gamma} = \langle A^{\mathrm{H}}_{\gamma,t}(t_0)\rangle^{t_0}_{\Psi_\gamma}. \tag{8.7}$$

So, the mean value of $A_\gamma(t)$ in a state $\Psi_\gamma(t)$ is equal to the mean value of $A^{\mathrm{H}}_{\gamma,t}(t_0)$ in the state $\Psi_\gamma(t_0)$. Taking into account that the only measurable (observable) physical quantities are the mean values [7, 9, 38], we infer that the descriptions of a quantum system along γ at a moment t through either one of the pairs $(\Psi_\gamma(t), A_\gamma(t))$ and $(\Psi_\gamma(t_0), A^{\mathrm{H}}_{\gamma,t}(t_0))$ are fully equivalent. The former one is the bundle Schrödinger picture of motion along γ, reviewed above in Sect. 8.1. The latter one is the *bundle Heisenberg picture of motion* of the quantum system along γ.[45] In it the time dependence of the bundle state vectors is entirely shifted to the observables in conformity with (8.6). In this description the bundle state vectors are constant and do not evolve in time. On the contrary, in it the observables depend on time and act on one and the same fibre of (F, π, M), the one to which belongs the (initial) bundle state vector. Their evolution is governed by the Heisenberg form of the bundle Schrödinger equation (6.25) which can be derived in the following way.

Substituting (5.10) and (7.1) into (8.6), we get

$$A^{\mathrm{H}}_{\gamma,t}(t_0) = l^{-1}_{\gamma(t_0)} \circ \mathcal{A}^{\mathrm{H}}_t(t_0) \circ l_{\gamma(t_0)} \colon F_{\gamma(t_0)} \to F_{\gamma(t_0)}, \tag{8.8}$$

where (cf. (8.6))

$$\mathcal{A}^{\mathrm{H}}_t(t_0) := \mathcal{U}(t_0, t) \circ \mathcal{A}(t) \circ \mathcal{U}(t, t_0) \colon \mathcal{F} \to \mathcal{F} \tag{8.9}$$

is the Heisenberg operator corresponding to $\mathcal{A}(t)$ in the Hilbert space description (see below).

A simple verification shows that

$$\mathrm{i}\hbar\frac{\partial\mathcal{A}^{\mathrm{H}}_t(t_0)}{\partial t} = \big[\mathcal{A}^{\mathrm{H}}_t(t_0), \mathcal{H}^{\mathrm{H}}_t(t_0)\big]_- + \mathrm{i}\hbar\Big(\frac{\partial\mathcal{A}}{\partial t}\Big)^{\mathrm{H}}_t(t_0). \tag{8.10}$$

Here $(\partial\mathcal{A}/\partial t)^{\mathrm{H}}_t(t_0)$ is obtained from (8.9) with $\partial\mathcal{A}/\partial t$ instead of $\mathcal{A}$ and

$$\mathcal{H}^{\mathrm{H}}_t(t_0) = \mathcal{U}^{-1}(t, t_0)\mathcal{H}(t)\mathcal{U}(t, t_0) = \mathrm{i}\hbar\mathcal{U}^{-1}(t, t_0)\frac{\partial\mathcal{U}(t, t_0)}{\partial t} \tag{8.11}$$

[45] Notice, the bundle Heisenberg picture is with respect to some (reference) path γ. We shell comment on this fact in Subsect. 8.2.3.

(cf. (8.9)) with $\mathcal{H}(t)$ being the usual Hamiltonian in $\mathcal{F}$ (see (2.9)), i.e. $\mathcal{H}^{\mathrm{H}}_t(t_0)$ is the Hamiltonian in the Heisenberg picture.

Finally, from (8.8) and (8.10), we obtain

$$\mathrm{i}\hbar\frac{\partial A^{\mathrm{H}}_{\gamma,t}(t_0)}{\partial t} = \left[A^{\mathrm{H}}_{\gamma,t}(t_0), H^{\mathrm{H}}_{\gamma,t}(t_0)\right]_- + \mathrm{i}\hbar\left(\frac{\partial\mathcal{A}}{\partial t}\right)^{\mathrm{H}}_{\gamma,t}(t_0) \tag{8.12}$$

in which all quantities with subscript γ are defined according to (8.8). This is the *bundle equation of motion (for the observables) in the Heisenberg picture* of motion of a quantum system. It determines the time evolution of the observables in this description.

8.2.2. Hilbert space introduction

Now we shall outline briefly how the above results can be obtained by transferring the conventional Heisenberg picture of motion from the Hilbert space $\mathcal{F}$ to its analogue in the Hilbert bundle (F,π,M).

The mathematical expectation of an observable $\mathcal{A}(t)$ in a state characterized by a bundle state vector $\psi(t)$ with a finite norm is (see (2.11), (2.5), and (5.4))

$$\langle\mathcal{A}(t)\rangle^t_\psi = \frac{\langle\psi(t)|\mathcal{A}(t)\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} = \frac{\langle\psi(t_0)|\mathcal{U}^{-1}(t,t_0)\mathcal{A}(t)\mathcal{U}(t,t_0)\psi(t_0)\rangle}{\langle\psi(t_0)|\psi(t_0)\rangle}.$$

Combining this with (8.9), we find:

$$\langle\mathcal{A}(t)\rangle^t_\psi = \langle\mathcal{A}^{\mathrm{H}}_t(t_0)\rangle^{t_0}_\psi = \langle\mathcal{A}^{\mathrm{H}}_t(t_0)\rangle^{t_0}_{\psi^{\mathrm{H}}_t}, \tag{8.13}$$

$$\psi^{\mathrm{H}}_t(t_0) := \psi(t_0). \tag{8.14}$$

Thus the pair $(\psi(t),\mathcal{A}(t))$ is equivalent to the pair $(\psi(t_0),\mathcal{A}^{\mathrm{H}}_t(t_0))$ from the view-point of observable quantities. The latter one realizes the Heisenberg picture in $\mathcal{F}$, i.e. in the Hilbert space description of quantum mechanics. In it the state vectors are constant while the observables, generally, change in time according to the Heisenberg form (8.10) of the equation of motion.

In the Hilbert bundle description to $\mathcal{A}$ and $\mathcal{A}^{\mathrm{H}}_t(t_0)$ correspond the quantities (see (7.1)), respectively, $A_\gamma(t) = l^{-1}_{\gamma(t)}\circ\mathcal{A}(t)\circ l_{\gamma(t)}$ and (see (8.9) and (5.10))

$$A^{\mathrm{H}}_{\gamma,t}(t_0) = l^{-1}_{\gamma(t_0)}\circ\mathcal{A}^{\mathrm{H}}_t(t_0)\circ l_{\gamma(t_0)} = U^{-1}_\gamma(t,t_0)\circ A_\gamma(t)\circ U_\gamma(t,t_0). \tag{8.15}$$

Hence to the Heisenberg operator $\mathcal{A}^{\mathrm{H}}_t$ corresponds exactly the above-introduced by (8.6) (Heisenberg) map $A^{\mathrm{H}}_{\gamma,t}(t_0)$. In particular, to the Hamiltonian $\mathcal{H}(t)$ and its Heisenberg form $\mathcal{H}^{\mathrm{H}}_t(t_0)$, given by (8.9) for $\mathcal{A}=\mathcal{H}$ or by (8.11) (cf. (2.9)), correspond the mappings (see (7.1) and (7.12)) $H_\gamma(t) = l^{-1}_{\gamma(t)}\circ\mathcal{H}(t)\circ l_{\gamma(t)}$ and (cf. (8.8) and (8.11))

$$H^{\mathrm{H}}_{\gamma,t}(t_0) = l^{-1}_{\gamma(t_0)}\circ\mathcal{H}^{\mathrm{H}}_t(t_0)\circ l_{\gamma(t_0)} = U^{-1}_\gamma(t,t_0)\circ H_\gamma(t)\circ U_\gamma(t,t_0) \tag{8.16}$$

the latter of which is exactly the one entering in (8.12).

Now it is a trivial verification that the mappings $A^{\mathrm{H}}_{\gamma,t}(t_0)$ satisfy the bundle Heisenberg equation of motion (8.12).

Thus the both approaches, Hilbert bundle and Hilbert space ones, are self-consistent and lead to one and the same final result, the bundle Heisenberg picture of motion.

8.2.3. Summary and inferences

According to the above results, in the bundle Heisenberg picture the state of a quantum system is represented by a time-independent bundle state vector

$$\Psi^{\mathrm{H}}_{\gamma,t}(t_0) = \Psi_\gamma(t_0) \in F_{\gamma(t_0)} \tag{8.17}$$

and every dynamical variable $\mathbb{A}$ is described by a time-depending mapping

$$A^{\mathrm{H}}_{\gamma,t}(t_0) := U^{-1}_\gamma(t,t_0) \circ A_\gamma(t) \circ U_\gamma(t,t_0) \in (\pi^M_0)^{-1}(\gamma(t_0)) \tag{8.18}$$

from the fibre over $\gamma(t_0)$ of the bundle $\mathrm{mor}_M(F,\pi,M) = (F^M_0, \pi^M_0, M)$ of point-restricted morphisms over M of (F,π,M).[46] Here $\gamma\colon J \to M$ is a path in the base M, $t \in J$ is arbitrary, and $t_0 \in J$ is arbitrarily fixed and interpreted as an initial moment at which the initial conditions determining the system's state and dynamical variables are supposed to be known.

By virtue of (8.7), (8.17), and (7.3), the mean values of a dynamical variable $\mathbb{A}$ are independent of the way of their calculation:

$$\langle A\rangle^{t,\gamma}_\Psi = \langle A^{\mathrm{H}}_{\gamma,t}(t_0)\rangle^{t_0}_{\Psi^{\mathrm{H}}_\gamma} = \langle \mathcal{A}\rangle^{t,\gamma}_\Psi = \langle \mathcal{A}^{\mathrm{H}}_t(t_0)\rangle^{t_0}_{\Psi^{\mathrm{H}}_\gamma}. \tag{8.19}$$

Hence the predictions of quantum mechanics are identical in the Hilbert bundle and Hilbert space descriptions, as well as in their presentations in the Schrödinger and Heisenberg pictures.

We want to emphasize on three features of the bundle Heisenberg picture as introduced above. First, in it the states are not represented via state liftings as in the Schrödinger picture, but by a particular bundle state vector corresponding to a concrete value of the state lifting of the reference path γ in the Schrödinger picture. Second, in it the dynamical variables are described via (time-dependent) mappings whose domain and range is the fibre over the same fixed point of the reference path γ in the system's Hilbert bundle, while in the Schrödinger picture the corresponding objects are liftings of paths in the bundle of restricted morphisms of the Hilbert bundle of states. Third, the bundle Heisenberg picture, as formulated above, is explicitly observer-dependent in a sense that it is always defined with respect to some reference path γ.[47] This fact is in contrast to the Schrödinger picture

[46] For the notation and mathematical details, see subsections 8.2.1 and 3.1.

[47] Such dependence exists also in the conventional Hilbert space description of quantum mechanics, but it is so deeply hidden that it seem not to be mentioned until now.

which is formulated in an observer-independent way, only in terms of liftings of paths and transports along paths in suitable bundles and in which the observer-independence is introduced via the initial conditions. Below an analogous description for the Heisenberg picture will be found too.

An interesting interpretation of the Heisenberg picture can be given in the bundle $\mathrm{mor}_M\,(F,\pi,M)=(F_0,\pi_0,M)$ of point-restricted morphisms over M of (F,π,M) (see subsection 8.1 or 3.1). Since U is a transport along paths in the bundle (F,π,M) of states, then, according to (3.47) (see also [59, equation (3.12)]), it induces a transport ${}^{\circ}U$ along paths in $\mathrm{mor}_M\,(F,\pi,M)$ whose action on a map $A_\gamma(s)\colon \pi^{-1}(\gamma(s))\to\pi^{-1}(\gamma(s))$ along $\gamma\colon J\to M$ is

$$ {}^{\circ}U_\gamma(t,s)(A_\gamma(s)) := U_\gamma(t,s)\circ A_\gamma(s)\circ U_\gamma(s,t)\in(\pi_0^M)^{-1}(\gamma(t)). \tag{8.20}$$

Comparing, from one hand, this definition with (8.6) and, from other hand, (8.17) with (5.7), we obtain respectively

$$ A^{\mathrm{H}}_{\gamma,t}(t_0) = {}^{\circ}U_\gamma(t_0,t)(A_\gamma(t)), \tag{8.21}$$

$$ \Psi^{\mathrm{H}}_{\gamma,t}(t_0) = U_\gamma(t_0,t)\Psi_\gamma(t). \tag{8.22}$$

Consequently the pair of transports $(U,{}^{\circ}U)$ along paths is just the mapping which maps the bundle Schrödinger picture into the bundle Heisenberg picture.

A simple corollary of (8.21) and (3.19) is that the Heisenberg operators $A^{\mathrm{H}}_{\gamma,t}(t_0)$ are connected by

$$ A^{\mathrm{H}}_{\gamma,t}(t_1) = {}^{\circ}U_{\gamma,t}(t_1,t_0)A^{\mathrm{H}}_{\gamma,t}(t_0) \tag{8.23}$$

for every initial moments $t_1,t_0\in J$. Obviously, the map $A^{\mathrm{H}}_{\gamma,t}\colon t_0\mapsto A^{\mathrm{H}}_{\gamma,t}(t_0)$ for every $t_0\in J$ is a lifting of γ from M to the bundle space of the bundle $\mathrm{mor}_M(F,\pi,M)$. Therefore the mapping $A^{\mathrm{H}}\colon\gamma\mapsto A^{\mathrm{H}}_{\gamma,t}$ is a lifting of paths in $\mathrm{mor}_M(F,\pi,M)$,

$$ A^{\mathrm{H}}\in\mathrm{PLift}(\mathrm{mor}_M(F,\pi,M)), \tag{8.24}$$

which, by virtue of (8.23) is ${}^{\circ}U$-transported along every path γ. Comparing (8.21) and (3.39), we infer that A^{H} coincides with the lifting ${}^{\circ}\overline{U}\in\mathrm{PLift}(\mathrm{mor}_M(F,\pi,M))$ generated by ${}^{\circ}U$ (see definition 3.5). This observation allows to be found a new form of the equations of motion for the observables in the Heisenberg picture which replaces the Schrödinger equation in it.

Let ${}^{\circ}D$ be the derivation along paths generated by the induced transport ${}^{\circ}U$ (see also [31,32]). In conformity with (3.47), we have ${}^{\circ}D\colon\gamma\mapsto{}^{\circ}D^\gamma\colon s\mapsto{}^{\circ}D^\gamma_s$ with

$$ {}^{\circ}D^\gamma_s(A) := \lim_{\varepsilon\to0}\Big\{\frac{1}{\varepsilon}\left[{}^{\circ}U(s,s+\varepsilon)\,(A_\gamma(s+\varepsilon))-A_\gamma(s)\right]\Big\}. \tag{8.25}$$

A simple calculation shows that in a local field of bases the matrix of ${}^\circ D^\gamma_s(A)$, in accordance with (3.49) is

$$\left[{}^\circ D^\gamma_s(A)\right] = -\left[\boldsymbol{A}_\gamma(s), \boldsymbol{\Gamma}_\gamma(s)\right]_- + \frac{\partial \boldsymbol{A}_\gamma(s)}{\partial s} \tag{8.26}$$

where $\boldsymbol{\Gamma}_\gamma(s) := \left[\Gamma^b{}_a(s;\gamma)\right] := \partial \boldsymbol{U}_\gamma(s,t)/\partial t|_{t=s}$ is the matrix of the coefficients of U (not of ${}^\circ U$!). From here, using (6.22) and (7.14), one, after some matrix algebra, finds the explicit form of (8.25):

$${}^\circ D^\gamma_t(A) = \frac{1}{\mathrm{i}\hbar}[A_\gamma(t), H_\gamma(t)]_- + \left(\frac{\partial \mathcal{A}}{\partial t}\right)_{\gamma(t)}, \tag{8.27}$$

where the last term is defined via (7.1) and H is the bundle Hamiltonian, given by (7.12).

The last result, together with (8.6), shows that the Heisenberg equation of motion (8.12) is equivalent to

$$\frac{\partial A^{\mathrm{H}}_{\gamma,t}(t_0)}{\partial t} = U_\gamma(t_0,t) \circ ({}^\circ D^\gamma_t(A)) \circ U_\gamma(t,t_0). \tag{8.28}$$

By the way, this equation is also an almost trivial corollary of (8.25), (8.21), and (3.19). But such a 'quick' derivation leaves the problem for the relation (equivalence) between equation (8.28) and 7.10 open.

Now the analogue of (6.28) is

$${}^\circ D^\gamma_t \circ (\overline{{}^\circ U}) = 0, \quad \overline{{}^\circ U}^\gamma(t_0,t_0) = \mathsf{id}_{\pi_0^{-1}(\gamma(t_0))}. \tag{8.29}$$

From here and (8.23), we derive the equation of motion as

$${}^\circ D^\gamma_{t_0}\left(A^{\mathrm{H}}\right) = 0 \tag{8.30}$$

which is another equivalent form of (8.12) or (8.28).

Since $\gamma\colon J \to M$ and $t \in J$ are arbitrary, the last equation is equivalent to

$${}^\circ D(A^{\mathrm{H}}) = 0. \tag{8.31}$$

This is the *bundle Heisenberg equation of motion* (for the observables) which replaces the bundle Schrödinger equation (6.27) in the Heisenberg picture. It does not depend on the reference path γ and, in this sense, is observer-independent. As in the Schrödinger picture (see Subsect. 8.1), here the observer-dependence is introduced by the initial conditions at some moment $t_0 \in J$. This is clearly seen from (8.29) regardless of the fact that the equation of this initial-value problem can be rewritten as

$${}^\circ D(\overline{U}) = 0 \tag{8.32}$$

which is independent of the reference path γ.

The Heisenberg bundle state vector (8.22) admits a treatment analogous to the one of $A^{\mathrm{H}}_{\gamma,t}(t_0)$. Indeed, define a lifting of paths

$$\Psi^{\mathrm{H}} \in \mathrm{PLift}(F, \pi, M) \tag{8.33}$$

by $\Psi^{\mathrm{H}}\colon \gamma \mapsto \Psi^{\mathrm{H}}_{\gamma,t}$ with $\Psi^{\mathrm{H}}_{\gamma,t}\colon t_0 \mapsto \Psi^{\mathrm{H}}_{\gamma,t}(t_0)$, $t_0 \in J$. By virtue of (8.21), this lifting is U-transported along every path γ, coincides with the lifting $\overline{U}$ generated by the evolution transport U (see definition 3.5), and, in conformity with (3.40) satisfies the equation

$$D(\Psi^{\mathrm{H}}) = 0 \tag{8.34}$$

with D being the derivation along paths generated by U. *Pro forma* the last equation coincides with the bundle Schrödinger equation (6.27) but its meaning is completely different: in the Heisenberg picture the system's state is described by a solution of (8.34) at a *single* point of some path, while in the Schrödinger picture the state is represented via the solution of (6.27) along a whole path, i.e. in the former case the state is given by a fixed bundle state vector, while in the latter one via a lifting of paths.

Now a brief comment on the beginning of Subsect. 8.2.1 is in order. It was shown that in a normal frame (8.4), described via some of the conditions (8.5), the matrix elements of an observable lifting of paths in the Schrödinger picture coincide with the ones in the Heisenberg picture, $\widetilde{(A(t))}_{ab} = (A^{\mathrm{H}}_{\gamma,t}(t_0))_{ab}$. In this frame the components of a bundle state vector $\Psi_\gamma(t)$ are $\widetilde{\Psi}^a_\gamma(t) = \big(U^{-1}_\gamma(t,t_0)\big)^a{}_b \Psi^b_\gamma(t) = \big(U_\gamma(t_0,t)\big)^a{}_b \Psi^b_\gamma(t) = \Psi^b_\gamma(t_0) = \big(\Psi^{\mathrm{H}}_{\gamma,t}(t_0)\big)^a$. These results can be expressed by the assertion that in a normal frame the Schrödinger picture of motion is identical with the Heisenberg one.

8.3. 'General' picture

The Schrödinger and Heisenberg pictures for describing a quantum system are not the only possible ones. Any transformation of the state vectors and observables preserving the scalar products leads to a new 'picture' . For investigating different problems, different such pictures may turn to be suitable. Below we present the general scheme by means of which such special representations of the quantum-mechanical motion are generated.

8.3.1. Introduction

The idea of a particular picture of motion is the simultaneous transformation of the (bundle) state vectors and the observables (observable liftings) in such a way that the scalar products remain unchanged. As a consequence of this, the physically predictable results of the theory are identical with the ones before the transformation. Formally one should proceed as follows.

Let V be a 'two-point' lifting of paths in $\mathrm{mor}_M(F,\pi,M)$, i.e. for every $\gamma\colon J\to M$, we have $V\colon\gamma\mapsto V_\gamma$ with $V_\gamma\colon(s,t)\mapsto V_\gamma(s,t)$ where $V_\gamma(s,t)\colon F_{\gamma(s)}\to F_{\gamma(t)}$.[48] Suppose the maps $V_\gamma(t,s)\colon F_{\gamma(s)}\to F_{\gamma(t)},\ s,t\in J$ are linear, of class C^1, and unitary, i.e. (see (3.12)) $V_\gamma^\ddagger(t,s)=V_\gamma^{-1}(s,t)$, where $V_\gamma^{-1}(s,t)$ is the *left* inverse of $V_\gamma(s,t)$.[49] A simple calculation shows that

$$\langle\Psi_\gamma(t)|\Psi_\gamma(t)\rangle_{\gamma(t)}=\langle\Psi^V_{\gamma,t}(t_1)|\Psi^V_{\gamma,t}(t_1)\rangle_{\gamma(t_1)} \tag{8.35}$$

$$\langle A_\gamma(t)\rangle^t_{\Psi_\gamma}=\langle A^V_{\gamma,t}(t_1)\rangle^{t_1}_{\Psi^V_{\gamma,t}}, \tag{8.36}$$

where (3.7) was used, $t_1\in J$, and

$$\Psi^V_{\gamma,t}(t_1):=V_\gamma(t_1,t)\Psi_\gamma(t)\in F_{\gamma(t_1)}, \tag{8.37}$$

$$A^V_{\gamma,t}(t_1):=V_\gamma(t_1,t)\circ A_\gamma(t)\circ V_\gamma^{-1}(t_1,t)\colon F_{\gamma(t_1)}\to F_{\gamma(t_1)}. \tag{8.38}$$

Thereof the pairs $(\Psi_\gamma(t),A_\gamma(t))$ and $(\Psi^V_{\gamma,t}(t_1),A^V_{\gamma,t}(t_1))$ provide a completely equivalent description of a given quantum system as the physical predictable results on their base are identical. The latter way of describing a quantum system will be called the *V-picture* or *general picture* of motion. For $t_1=t$ and $V_\gamma(t,t)=\mathsf{id}_{F_{\gamma(t)}}$ it coincides with the Schrödinger picture and for $t_1=t_0$ and $V_\gamma(t_0,t)=U_\gamma(t_0,t)$ it reproduces the Heisenberg picture.

The analogues of (8.36), (8.37), and (8.38) in the Hilbert space space description in the Hilbert space $\mathcal{F}$, which is the typical fibre of the Hilbert bundle (F,π,M) of states, are respectively:

$$\langle\mathcal{A}(t)\rangle^t_\psi=\langle\mathcal{A}^V_t(t_1)\rangle^{t_1}_{\psi^V_t}\big(=\langle A_\gamma(t)\rangle^t_{\Psi_\gamma}\big), \tag{8.39}$$

$$\psi^{\mathcal{V}}_t(t_1):=\mathcal{V}(t_1,t)\psi(t)\in\mathcal{F}, \tag{8.40}$$

$$\mathcal{A}^{\mathcal{V}}_t(t_1):=\mathcal{V}(t_1,t)\circ\mathcal{A}(t)\circ\mathcal{V}^{-1}(t_1,t)\colon\mathcal{F}\to\mathcal{F} \tag{8.41}$$

where $\mathcal{V}(t_1,t)\colon\mathcal{F}\to\mathcal{F}$ is the linear and unitary, i.e. $\mathcal{V}^\dagger(t_1,t)=(\mathcal{V}(t,t_1))^{-1}$, operator corresponding to the map $V(t_1,t)\colon F_{\gamma(t)}\to F_{\gamma(t_1)}$ via (cf. (5.10))

$$V_\gamma(t_1,t)=l^{-1}_{\gamma(t_1)}\circ\mathcal{V}(t_1,t)\circ l_{\gamma(t)}. \tag{8.42}$$

The description of the quantum evolution in the Hilbert space $\mathcal{F}$ via $\psi^{\mathcal{V}}_t(t_1)$ and $\mathcal{A}^{\mathcal{V}}_t(t_1)$ is the *V-picture* of motion in $\mathcal{F}$. Besides, due to (4.3), (7.1), and (8.37)–(8.42), the following relations are valid:

$$\Psi^{\mathcal{V}}_{\gamma,t}(t_1):=l^{-1}_{\gamma(t_1)}\left(\psi^{\mathcal{V}}_t(t_1)\right)=\Psi^V_{\gamma,t}(t_1), \tag{8.43}$$

$$A^{\mathcal{V}}_{\gamma,t}(t_1):=l^{-1}_{\gamma(t_1)}\circ\mathcal{A}^{\mathcal{V}}_t(t_1)\circ l_{\gamma(t_1)}=A^V_{\gamma,t}(t_1). \tag{8.44}$$

[48] Example of such map V is a transport along paths in $\mathrm{mor}_M(F,\pi,M)$.

[49] Every unitary (and hence Hermitian) linear transport along paths in $\mathrm{Mor}_M(F,\pi,M)$ provides an example of V with the required properties. In particular, for V can be taken the transport ${}^\circ L$ associated to some unitary linear transport L along paths in (F,π,M). The choice $L=U$, U being the evolution transport, returns us to the Heisenberg picture — *vide infra*.

According to (8.42)–(8.44), the sets of equalities (8.36)–(8.38) and (8.39)–(8.41) are equivalent; they are, respectively, the Hilbert bundle and the (usual) Hilbert space descriptions of the V-picture of motion.

8.3.2. Equations of motion

The equations of motion in the V-picture cannot be obtained directly by differentiating (8.37) and (8.38) with respect to t because derivatives like $\partial V_\gamma(t_1,t)/\partial t$ are not ('well') defined due to $V_\gamma(t_1,t)\colon F_{\gamma(t)} \to F_{\gamma(t_1)}$. They can be derived by differentiating the corresponding to (8.37) and (8.38) matrix equations, but below we shall describe another method which explicitly reveals the connections between the conventional and the bundle descriptions of quantum evolution. The easiest way for deriving the equations of motion in the V-picture, is to transform the conventional Schrödinger equations (by means of (8.40)) into the V-picture and then to transform the obtained equations into their bundle versions. With respect to the observables, a procedure similar to the one of Subsect. 8.2.1 should be followed.

Differentiating (8.40) with respect to t, substituting into the so-obtained result the Schrödinger equation (2.6), and introducing the modified Hamiltonian

$$\widetilde{\mathcal{H}}(t) := \mathcal{H}(t) - {}_{\mathcal{V}}\mathcal{H}(t_1,t), \tag{8.45}$$

$${}_{\mathcal{V}}\mathcal{H}(t_1,t) := \mathrm{i}\hbar\frac{\partial\mathcal{V}^{-1}(t_1,t)}{\partial t}\circ\mathcal{V}(t_1,t) = -\mathrm{i}\hbar\mathcal{V}^{-1}(t_1,t)\circ\frac{\partial\mathcal{V}(t_1,t)}{\partial t}, \tag{8.46}$$

we find the *equation of motion* for the state vectors in the V-picture as

$$\mathrm{i}\hbar\frac{\partial\psi_t^{\mathcal{V}}(t_1)}{\partial t} = \widetilde{\mathcal{H}}_t^{\mathcal{V}}(t_1)\psi_t^{\mathcal{V}}(t_1). \tag{8.47}$$

Here

$$\widetilde{\mathcal{H}}_t^{\mathcal{V}}(t_1) = \mathcal{V}(t_1,t)\circ\widetilde{\mathcal{H}}(t)\circ\mathcal{V}^{-1}(t_1,t) = \mathcal{H}_t^{\mathcal{V}}(t_1) - {}_{\mathcal{V}}\mathcal{H}_t^{\mathcal{V}}(t_1), \tag{8.48}$$

where

$$\begin{aligned}\mathcal{H}_t^{\mathcal{V}}(t_1) &:= \mathcal{V}(t_1,t)\circ\mathcal{H}(t)\circ\mathcal{V}^{-1}(t_1,t),\\ {}_{\mathcal{V}}\mathcal{H}_t^{\mathcal{V}}(t_1) &:= \mathcal{V}(t_1,t)\circ_{\mathcal{V}}\mathcal{H}(t_1,t)\circ\mathcal{V}^{-1}(t_1,t) = -\mathrm{i}\hbar\frac{\partial\mathcal{V}(t_1,t)}{\partial t}\circ\mathcal{V}^{-1}(t_1,t),\end{aligned} \tag{8.49}$$

is the V-form of (8.45).

The *equation of motion for the observables in the V-picture* in $\mathcal{F}$ is obtained in an analogous way. Differentiating (8.41) with respect to t and applying (8.49), we find

$$\mathrm{i}\hbar\frac{\partial\mathcal{A}_t^{\mathcal{V}}(t_1)}{\partial t} = \left[\mathcal{A}_t^{\mathcal{V}}(t_1),{}_{\mathcal{V}}\mathcal{H}_t^{\mathcal{V}}(t_1)\right]_- + \mathrm{i}\hbar\left(\frac{\partial\mathcal{A}(t)}{\partial t}\right)_t^{\mathcal{V}}(t_1). \tag{8.50}$$

The bundle equations of motion in the V-picture are corollaries of the already obtained ones in $\mathcal{F}$. In fact, differentiating the first equalities from (8.43) and (8.44) with respect to t and then using (8.47), (8.50), (8.43), and (8.44), we, respectively, get:

$$\mathrm{i}\hbar\frac{\partial\Psi^V_{\gamma,t}(t_1)}{\partial t} = \widetilde{H}^V_{\gamma,t}(t_1)\Psi^V_{\gamma,t}(t_1), \tag{8.51}$$

$$\mathrm{i}\hbar\frac{\partial A^V_{\gamma,t}(t_1)}{\partial t} = \left[A^V_{\gamma,t}(t_1), {}_V H^V_{\gamma,t}(t_1)\right]_- + \mathrm{i}\hbar\left(\frac{\partial\mathcal{A}(t)}{\partial t}\right)^V_{\gamma,t}(t_1). \tag{8.52}$$

Here

$$\begin{aligned}\widetilde{H}^V_{\gamma,t}(t_1) &= l^{-1}_{\gamma(t_1)}\circ\widetilde{\mathcal{H}}^{\mathcal{V}}_t(t_1)\circ l_{\gamma(t_1)} = V_\gamma(t_1,t)\circ\widetilde{H}_\gamma(t)\circ V^{-1}_\gamma(t_1,t),\\ {}_V H^V_{\gamma,t}(t_1) &= l^{-1}_{\gamma(t_1)}\circ_{\mathcal{V}}\mathcal{H}^{\mathcal{V}}_t(t_1)\circ l_{\gamma(t_1)} = V_\gamma(t_1,t)\circ_{\mathcal{V}} H_\gamma(t_1,t)\circ V^{-1}_\gamma(t_1,t),\end{aligned} \tag{8.53}$$

where $\widetilde{H}_\gamma(t) := l^{-1}_{\gamma(t)}\circ\widetilde{\mathcal{H}}(t)\circ l_{\gamma(t)}$ and ${}_{\mathcal{V}}H_\gamma(t_1,t) = -\mathrm{i}\hbar V^{-1}(t_1,t)\circ l_{\gamma(t_1)}\circ\frac{\partial\mathcal{V}(t_1,t)}{\partial t}\circ l_{\gamma(t)}$, are, respectively, the modified and 'additional' Hamiltonians in the V-picture (cf. (8.45), (8.38), and (8.44)).

8.3.3. Evolution operator and transport

In the V-picture the evolution operator $\mathcal{U}^{\mathcal{V}}$ in $\mathcal{F}$ and evolution transport U^V in (F,π,M) are define, respectively, by (cf. (2.1) and (5.7))

$$\psi^{\mathcal{V}}_t(t_1) = \mathcal{U}^{\mathcal{V}}(t,t_1,t_0)\psi^{\mathcal{V}}_{t_0}(t_1), \tag{8.54}$$

$$\Psi^V_{\gamma,t}(t_1) = U^V_\gamma(t,t_1,t_0)\Psi^V_{\gamma,t_0}(t_1). \tag{8.55}$$

Due to (8.47) and (8.51), they satisfy the following initial-value problems:

$$\mathrm{i}\hbar\frac{\partial\mathcal{U}^{\mathcal{V}}(t,t_1,t_0)}{\partial t} = \widetilde{\mathcal{H}}^{\mathcal{V}}_t(t_1)\circ\mathcal{U}^{\mathcal{V}}(t,t_1,t_0), \qquad \mathcal{U}^{\mathcal{V}}(t_0,t_1,t_0) = \mathsf{id}_{\mathcal{F}}, \tag{8.56}$$

$$\mathrm{i}\hbar\frac{\partial U^V_\gamma(t,t_1,t_0)}{\partial t} = \widetilde{H}^V_{\gamma,t}(t_1)\circ U^V_\gamma(t,t_1,t_0), \quad U^V_\gamma(t_0,t_1,t_0) = \mathsf{id}_{F_{\gamma(t_1)}}. \tag{8.57}$$

The relations between the evolution operator or evolution transport in the Schrödinger picture and V-picture can be found as follows. From one hand, combining (8.54) , (8.40), and (2.1) and, from another hand, using (8.55), (8.37), and (5.7), we respectively obtain:

$$\mathcal{U}^{\mathcal{V}}(t,t_1,t_0) = \mathcal{V}(t_1,t)\circ\mathcal{U}(t,t_0)\circ\mathcal{V}^{-1}(t_1,t_0)\colon\mathcal{F}\to\mathcal{F}, \tag{8.58}$$

$$U^V_\gamma(t,t_1,t_0) = V_\gamma(t_1,t)\circ U_\gamma(t,t_0)\circ V^{-1}_\gamma(t_1,t_0)\colon F_{\gamma(t_1)}\to F_{\gamma(t_1)}. \tag{8.59}$$

Notice, in the Heisenberg picture, we have

$$\mathcal{U}^{\mathrm{H}}(t,t_0,t_0) = \mathsf{id}_{\mathcal{F}}, \qquad U^{\mathrm{H}}_\gamma(t,t_0,t_0) = \mathsf{id}_{F_{\gamma(t_0)}}. \tag{8.60}$$

Substituting in (8.59) the equalities (8.42) and (5.10) and taking into account (8.58), we find the connection between the evolution operator and transport in the V-picture as

$$U_\gamma^V(t,t_1,t_0) = l^{-1}_{\gamma(t_1)} \circ \mathcal{U}^{\mathcal{V}}(t,t_1,t_0) \circ l_{\gamma(t_1)}. \tag{8.61}$$

8.3.4. Interaction interpretation

The derived here equations of motion have a direct practical applications in connection with the approximate treatment of the problem of quantum evolution of state vectors and observables (cf. [9, ch. VIII, § 14]). Indeed, by (8.45) is fulfilled $\mathcal{H}(t) =_{\mathcal{V}} \mathcal{H}(t,t_1) + \widetilde{\mathcal{H}}(t)$. We can consider

$$\mathcal{H}^{(0)}(t) :=_{\mathcal{V}} \mathcal{H}(t_1,t) = \mathrm{i}\hbar \frac{\partial \mathcal{V}^{-1}(t_1,t)}{\partial t} \circ \mathcal{V}(t_1,t) \tag{8.62}$$

as a given approximate (unperturbed) Hamiltonian of the quantum system with evolution operator $\mathcal{U}^{(0)}(t_1,t) = \mathcal{V}(t_1,t)$. (In this case $H^{(0)}(t)$ is independent of t_1 and $\mathcal{V}^{-1}(t_1,t) = \mathcal{V}(t,t_1)$.) Then $\widetilde{\mathcal{H}}(t)$ may be regarded, in some 'good' cases, as a 'small' correction to $H^{(0)}(t)$. In other words, we can say that $H^{(0)}(t)$ is the Hamiltonian of the 'free' system, while $\mathcal{H}(t)$ is its Hamiltonian when a given interaction with Hamiltonian $\widetilde{\mathcal{H}}(t)$ is introduced.

In this interpretation the V-picture is the well known *interaction picture*. In it one supposes to be given the basic (zeroth order) Hamiltonian $\mathcal{H}^{(0)}(t) := {}_{\mathcal{V}}\mathcal{H}(t,t_1)$ and the interaction Hamiltonian $\mathcal{H}^{(I)}(t) = \widetilde{\mathcal{H}}(t)$. On their base can be computed all other quantities of the system described by them. In particular, all of the above results hold true for $\mathcal{V}(t_1,t) = \mathcal{U}^{(0)}(t_1,t) = \mathrm{Texp}\Big(\int\limits_t^{t_1} \mathcal{H}^{(0)}(\tau)d\tau/\mathrm{i}\hbar\Big)$. Besides, in this case the total evolution operator $\mathcal{U}(t,t_0) = \mathrm{Texp}\Big(\int\limits_{t_0}^{t} \mathcal{H}(\tau)d\tau/\mathrm{i}\hbar\Big)$ splits into

$$\mathcal{U}(t,t_0) = \mathcal{U}^{(0)}(t,t_0) \circ \mathcal{U}^{(I)}(t,t_0) \tag{8.63}$$

with $\mathcal{U}^{(I)}(t,t_0) := \mathrm{Texp}\Big(\int\limits_{t_0}^{t} \big(\mathcal{H}^{(I)}\big)_\tau^{\mathcal{U}^{(0)}(t_0)}\, d\tau/\mathrm{i}\hbar\Big)$, where $\big(\mathcal{H}^{(I)}\big)_\tau^{\mathcal{U}^{(0)}(t_0)}$ is an operator given by (8.48) for $\widetilde{\mathcal{H}} = \mathcal{H}^{(I)}$, $t_1 = t_0$, and $\mathcal{V}(t_0,t) = \mathcal{U}^{(0)}(t_0,t)$. Now the equations of motion (8.47) and (8.50) take, respectively, the form:

$$\mathrm{i}\hbar \frac{\partial \psi^{(I)}(t)}{\partial t} = \Big(\mathcal{H}^{(I)}\Big)^t_{\mathcal{U}^{(0)}}(t_0)\psi^{(I)}(t), \tag{8.64}$$

$$\mathrm{i}\hbar \frac{\partial \mathcal{A}^{(I)}(t)}{\partial t} = \Big[\mathcal{A}^{(I)}(t), \Big(\mathcal{H}^{(I)}\Big)^t_{\mathcal{U}^{(0)}}(t_0)\Big]_- + \mathrm{i}\hbar \left(\frac{\partial \mathcal{A}}{\partial t}\right)^{\mathcal{U}^{(0)}}_t (t_0) \tag{8.65}$$

where $\psi^{(I)}(t) := \psi_t^{\mathcal{U}^{(0)}}(t_0)$ and $\mathcal{A}^{(I)}(t) := \mathcal{A}_t^{\mathcal{U}^{(0)}}(t_0)$. Up to notation, the last two equations coincide respectively with equations (55) and (56) of [9, ch.VIII, § 15].

The bundle form of the interaction interpretation of the V-picture of motion will not be presented here as an almost evident one.

8.3.5. Some inferences

Partially the conclusions of Subsect. 8.2.3 are valid *mutatis mutandis* in the general V-picture of motion. In short, their essence is the following.

In the bundle V-picture the system's state is represented by a, generally, time-dependent bundle state vector (8.37) from a fixed fibre over the reference path γ. The dynamical variables are described via, generally, time-depending maps acting on this single fibre. Due to (8.29), the Schrödinger and the V-picture are identical from the view-point of predicting physical results.

If V happens to be a (Hermitian linear) transport along paths in $\mathcal{F}$, then the whole concluding part of Subsect. 8.2.3, beginning with the paragraph containing equation (8.20), is valid in the case of the V-picture provided V is taken for $^{\circ}U$, Ψ^V for Ψ^{H}, and by D is understood the generated by V derivation along paths. (The particular choice $V = {}^{\circ}U$ reduces the V-picture to the Heisenberg one.) But if V is not a transport along paths, the conclusions from the last part of Subsect. 8.2.3 cannot be applied.

9. Integrals of motion

The integrals of motion, called also constants of motion, are the quantum-mechanical analogues of the preserved quantities in classical physics [63, chapter V, §§ 19, 20; chapter VIII, § 12]. They provide invariant characteristics of a quantum system which do not change in time. An important example of this kind is the energy of a system with explicitly time-independent Hamiltonian. In more special cases, such quantities are the angular momentum, parity, etc. The aim of this section is the development of the general formalism of integrals of motion in the bundle version of quantum mechanics.

9.1. Hilbert space description

Usually [9, ch. VIII, § 12], [7, § 28] an *explicitly not depending on time* dynamical variable is called an integral (or a constant) of motion if the corresponding to it observable is time-independent in the Heisenberg picture of motion. Due to (8.10) this means

$$0 = \mathrm{i}\hbar\frac{\partial\mathcal{A}_t^{\mathrm{H}}(t_0)}{\partial t} = \left[\mathcal{A}_t^{\mathrm{H}}(t_0), \mathcal{H}_t^{\mathrm{H}}(t_0)\right]_{-}. \tag{9.1}$$

Hence, if $\partial\mathcal{A}(t)/\partial t = 0$, then $\mathcal{A}$ is an integral of motion if and only if it commutes with the Hamiltonian. By virtue of (8.9), (8.41), and (8.50), this result is true in any picture of motion.

If (9.1) holds, then $\partial\mathcal{A}(t)/\partial t = 0$ and (8.9) imply

$$\mathcal{A}(t) = \mathcal{A}(t_0) = \mathcal{A}_t^{\mathrm{H}}(t_0) = \mathcal{A}_{t_0}^{\mathrm{H}}(t_0). \tag{9.2}$$

From (8.9) and (9.2) one easily obtains that (9.1) (under the assumption $\partial\mathcal{A}(t)/\partial t = 0$) is equivalent to the commutativity of the observable and the evolution operator:

$$[\mathcal{A}(t_0), \mathcal{U}(t_0, t)]_- = 0 \tag{9.3}$$

which, in connection with further generalizations, is better to be written as

$$\mathcal{A}(t_0) \circ \mathcal{U}(t_0, t) = \mathcal{U}(t_0, t) \circ \mathcal{A}(t). \tag{9.3$'$}$$

It is almost evident that the mean values of the integrals of motion are constant:

$$\langle\mathcal{A}(t)\rangle_\psi^t = \langle\mathcal{A}_t^{\mathrm{H}}(t_0)\rangle_\psi^{t_0} = \langle\mathcal{A}_{t_0}^{\mathrm{H}}(t_0)\rangle_\psi^{t_0} = \langle\mathcal{A}(t_0)\rangle_\psi^{t_0}. \tag{9.4}$$

In particular, if $\psi^{\mathrm{H}}(t) = \psi(t_0)$ is an eigenvector of $\mathcal{A}_t^{\mathrm{H}}(t_0)$ with eigenvalue a, i.e. $\mathcal{A}_t^{\mathrm{H}}(t_0)\psi^{\mathrm{H}}(t) = a\psi^{\mathrm{H}}(t)$, then $a = \mathrm{const}$ as $\langle\mathcal{A}_t^{\mathrm{H}}(t_0)\rangle_{\psi^{\mathrm{H}}}^{t_0} = a$. Besides, in the Schrödinger picture we have $\mathcal{A}(t_0)\psi(t) = a\psi(t)$.

Evidently, the identity map $\mathsf{id}_{\mathcal{F}}$, which plays the rôle of the unit operator in $\mathcal{F}$, is an integral on motion. For it every state vector is an eigenvector with $1 \in \mathbb{R}$ as eigenvalue.

Now we shall generalize the above material in the case when $\partial\mathcal{A}(t)/\partial t$ may be different from zero.

Definition 9.1. A dynamical variable, which may be explicitly time-dependent, is an integral (or a constant) of motion if the mean values of the corresponding to it observable are time-independent.

According to (7.3), (8.36) and (8.39) this definition does not depend on the used concrete picture of motion. Hence, without a lost of generality, we consider at first the Schrödinger picture in $\mathcal{F}$.

So, by definition, $\mathcal{A}(t)\colon \mathcal{F} \to \mathcal{F}$ is an integral of motion if

$$\langle\mathcal{A}(t)\rangle_\psi^t = \langle\mathcal{A}(t_0)\rangle_\psi^{t_0} \tag{9.5}$$

for some given instant of time t_0.

Due to (2.11), (2.1), (2.5), and (8.9) the last equation is equivalent to

$$\mathcal{A}(t) = \mathcal{U}(t, t_0) \circ \mathcal{A}(t_0) \circ \mathcal{U}(t_0, t) = \mathcal{A}_{t_0}^{\mathrm{H}}(t) \tag{9.6}$$

or to

$$\mathcal{U}(t_0,t)\circ\mathcal{A}(t)=\mathcal{A}(t_0)\circ\mathcal{U}(t_0,t). \tag{9.7}$$

Thus (9.3′) remains true in the general case, when it generalizes the commutativity of an observable and the evolution operator; in fact, in this case we can say, by definition, that $\mathcal{A}$ and $\mathcal{U}$ commute iff (9.7) holds.

Differentiating (9.6) with respect to t and using (2.9), we see that $\mathcal{A}$ is an integral of motion iff

$$\mathrm{i}\hbar\frac{\partial\mathcal{A}(t)}{\partial t}+[\mathcal{A}(t),\mathcal{H}(t)]_{-}=0. \tag{9.8}$$

For $\partial\mathcal{A}(t)/\partial t=0$ this equation reduces to (9.1). Indeed, according to equations (8.9) and (8.10), in the Heisenberg picture (9.8) is equivalent to

$$0=\mathrm{i}\hbar\left(\frac{\partial\mathcal{A}(t)}{\partial t}\right)^{\mathrm{H}}_{t}(t_0)+\left[\mathcal{A}^{\mathrm{H}}_t(t_0),\mathcal{H}^{\mathrm{H}}_t(t_0)\right]_{-}=\mathrm{i}\hbar\frac{\partial\mathcal{A}^{\mathrm{H}}_t(t_0)}{\partial t} \tag{9.9}$$

which proves our assertion. Besides, from (9.9) follows

$$\mathcal{A}^{\mathrm{H}}_t(t_0)=\mathcal{A}^{\mathrm{H}}_{t_0}(t_0)=\mathcal{A}(t_0) \tag{9.10}$$

but now $\mathcal{A}(t_0)$ is generally different from $\mathcal{A}(t)$. In this way we have proved that an *observable is an integral of motion iff in the Heisenberg picture it coincides with its initial value in the Schrödinger picture.*

Analogously to the explicitly time-independent case considered above, now one can easily prove that, if some state vector is an eigenvector for $\mathcal{A}$ with an eigenvalue a, then $\mathcal{A}$ is an integral of motion iff a is time-independent, i.e. $a=\mathrm{const}$.[50]

9.2. Hilbert bundle description

The next definition is a bundle version of definition 9.1.

Definition 9.1′. A dynamical variable is called an integral of motion if the corresponding to it observable lifting has time-independent mean values.

So, if A is the lifting of paths corresponding to a dynamical variable $\mathbb{A}$ (see Sect. 7), then $\mathbb{A}$ (or A) is an integral of motion iff

$$\langle A\rangle^{t,\gamma}_{\Psi}:=\langle A_\gamma(t)\rangle^{t}_{\Psi_\gamma}=\langle A_\gamma(t_0)\rangle^{t_0}_{\Psi_\gamma}=:\langle A\rangle^{t_0,\gamma}_{\Psi} \tag{9.11}$$

which, due to (7.3) is equivalent (and equal) to (9.5). Consequently, definitions 9.1 and 9.1′ are equivalent: $\mathbb{A}$ is an integral of motion in the Hilbert

[50] In fact, in this case we have $\mathcal{A}(t)\psi(t)=a(t)\psi(t)$ for $\psi(t)$ satisfying $\mathrm{i}\hbar\frac{d\psi(t)}{dt}=\mathcal{H}(t)\psi(t)$. The integrability condition for this system of equations (with respect to $\psi(t)$) is $\mathrm{i}\hbar\frac{\partial\mathcal{A}(t)}{\partial t}+[\mathcal{A}(t),\mathcal{H}(t)]_{-}=\mathrm{i}\hbar\frac{da(t)}{dt}\,\mathsf{id}_{\mathcal{F}}$ from where the above result follows.

space description iff it is such in the Hilbert bundle one. Therefore we can simply say that a dynamical variable is integral of motion if its mean values are time-independent.

From (9.6), (5.7), (7.2), (3.7), (5.14), (8.15), and (8.20), we see that equation (9.11) is equivalent to

$$A_\gamma(t) = U_\gamma(t,t_0)\circ A_\gamma(t_0)\circ U_\gamma(t_0,t) = {}^{\mathcal{C}}U_\gamma(t,t_0)(A(t_0)) = A^{\mathrm{H}}_{\gamma,t_0}(t) \quad (9.12)$$

where ${}^{\mathcal{C}}U$is the associated to U transport in $\mathrm{mor}_M(F,\pi,M)$.

A feature of the Hilbert bundle description is that in it, for the difference of the Hilbert space one, we cannot directly differentiate with respect to t maps like $A_\gamma(t)\colon F_{\gamma(t)} \to F_{\gamma(t)}$ and $U_\gamma(t,t_0)\colon F_{\gamma(t_0)} \to F_{\gamma(t)}$. So, to obtain the differential form of (9.12) (or (9.11)), we differentiate with respect to t the matrix form of (9.12) in a given field of bases (see Sect. 6). Thus, using (6.18), we find

$$\mathrm{i}\hbar\frac{\partial \boldsymbol{A}_\gamma(t)}{\partial t} + \big[\boldsymbol{A}_\gamma(t), \boldsymbol{H}^{\mathbf{m}}_\gamma(t)\big]_- = 0. \quad (9.13)$$

Because of (6.22) and (6.32), this equation is the local matrix form of the invariant equation[51]

$$\big(\widetilde{D}^\gamma_t(A)\big)(\Psi) = 0 \quad (9.14)$$

for every state lifting Ψ.

Consequently a *dynamical variable is an integral of motion iff the induced derivative along paths of the corresponding to it observable lifting has a vanishing action on the state liftings.*

If in some basis $\boldsymbol{A}_\gamma(t) = \mathrm{const} = \boldsymbol{A}_\gamma(t_0)$, then, with the help of (9.8), we get $[\boldsymbol{A}_\gamma(t), \boldsymbol{H}^{\mathbf{m}}_\gamma(t)]_- = 0$, i.e. the matrix of A_γ and the matrix-bundle Hamiltonian commute. It is important to note, from here does not follow the commutativity of the maps $A_\gamma(t)$, representing an observable by (7.1), and the bundle Hamiltonian (7.12) because the matrix of the latter is connected with the matrix-bundle Hamiltonian through (7.14).

If the bundle state vector $\Psi_\gamma(t)$ is an eigenvector for $A_\gamma(t)$, that is $A_\gamma(t)\Psi_\gamma(t) = a(t)\Psi_\gamma(t),\ a(t)\in\mathbb{R}$, then $\langle A_\gamma(t)\rangle^t_{\Psi_\gamma} = a(t)$. Hence from (9.6) follows that A_γ is an integral of motion iff $a(t) = \mathrm{const} = a(t_0)$.

Rewriting equation (9.13) in the form of Lax pair equation [64]

$$\frac{\partial}{\partial t}\boldsymbol{A}_\gamma(t) = -\frac{1}{\mathrm{i}\hbar}[\boldsymbol{A}_\gamma(t), \boldsymbol{\mathcal{H}}^{\mathbf{m}}_\gamma(t)]_- = [\boldsymbol{A}_\gamma(t), \boldsymbol{\Gamma}_\gamma(t)]_-, \quad (9.15)$$

where (6.22) was taken into account, we see that $\mathbb{A}$ *is an integral of motion iff in some (and hence in any) field of bases the matrices* $\boldsymbol{A}_\gamma(t)$ *and* $\boldsymbol{\Gamma}_\gamma(t)$ *form a Lax pair.*

[51] Recall, the induced derivation $\widetilde{D}$ along paths was defined via (3.35)–(3.37).

It is known [65, sect. 2] that the Lax pair equation (9.15) is invariant under transformations of a form

$$\boldsymbol{A}_\gamma(t) \mapsto \boldsymbol{W}\boldsymbol{A}_\gamma(t)\boldsymbol{W}^{-1}, \qquad \boldsymbol{\Gamma}_\gamma(t) \mapsto \boldsymbol{W}\boldsymbol{\Gamma}_\gamma(t)\boldsymbol{W}^{-1} - \frac{\partial \boldsymbol{W}}{\partial t}\boldsymbol{W}^{-1} \tag{9.16}$$

where $\boldsymbol{W}$ is a nondegenerate matrix, possibly depending on γ and t in our case. Hence $\boldsymbol{A}_\gamma(t)$ transforms as a tensor while $\boldsymbol{\Gamma}_\gamma(t)$ transforms as the matrix of the coefficients of a linear connection. These observations fully agree with our results of Sect. 6, expressed by equations (6.5) and (6.23) with $(\boldsymbol{\Omega}^\top(t;\gamma))^{-1} = \boldsymbol{W}$, and give independent arguments for treating (up to a constant) the matrix-bundle Hamiltonian as a gauge (connection) field.

Another invariant bundle necessary and sufficient condition for a dynamical variable to be an integral of motion can be found as follows. In the Heisenberg picture (9.11) transforms to

$$\langle A^{\mathrm{H}}_{\gamma,t}(t_0)\rangle^{t_0}_{\Psi_\gamma} = \langle A^{\mathrm{H}}_{\gamma,t_0}(t_0)\rangle^{t_0}_{\Psi_\gamma} \tag{9.17}$$

which is equivalent to (cf. (9.10))

$$A^{\mathrm{H}}_{\gamma,t}(t_0) = A^{\mathrm{H}}_{\gamma,t_0}(t_0) \ \ (= A_\gamma(t_0))\,. \tag{9.18}$$

So, due to (8.12), an observable lifting A is an integral of motion if and only if (cf. (9.9))

$$0 = \mathrm{i}\hbar\frac{\partial A^{\mathrm{H}}_{\gamma,t}(t_0)}{\partial t} = \left[A^{\mathrm{H}}_{\gamma,t}(t_0), H^{\mathrm{H}}_{\gamma,t}(t_0)\right]_- + \mathrm{i}\hbar\left(\frac{\partial \mathcal{A}}{\partial t}\right)^{\mathrm{H}}_{\gamma,t}(t_0). \tag{9.19}$$

This equation, according to (8.28), is equivalent to (cf. (9.14))

$${}^\circ D^\gamma_t(A) = 0. \tag{9.20}$$

Since γ and t are arbitrary, we can rewrite the last equation as

$${}^\circ D(A) = 0. \tag{9.21}$$

We can paraphrase this result by saying that a dynamical variable is an integral of motion iff the corresponding to it observable lifting is linearly transported (along paths) by means of the transport ${}^\circ U$ associated to the evolution transport U. The last result is explicitly expressed by (9.12).

Therefore a *dynamical variable is an integral of motion iff the corresponding to it observable lifting of paths in* $\mathrm{mor}_M(F,\pi,M)$ *has a vanishing derivative with respect to the derivation along paths in* $\mathrm{mor}_M\,(F,\pi,M)$ *induces by the evolution transport.* According to definition 3.5 (see also (3.39) and (3.40)) and equations (9.12) and (9.20), the same result can be expressed by saying that a *dynamical variable is an integral of motion iff the corresponding to it observable lifting is a lifting of paths generated by the*

evolution transport. Paraphrasing (see (9.12)), we can also assert that a dynamical variable is an integral of motion iff the corresponding to it observable lifting of paths is $^{\mathcal{C}}U$-transported along the paths in the base M of the bundle $\mathrm{mor}_M(F,\pi,M)$.

Ending, we notice that the descriptions of the integrals of motion in the Hilbert space $\mathcal{F}$ and in the Hilbert bundle (F,π,M) are completely equivalent because of (7.1) and (4.1).

10. Mixed states

The most general description of the state of a quantum system in the framework of standard quantum mechanics is provide by the so-called density (or statistical) operator (or matrix). It allows a uniform description of pure and mixed states. At a physical degree of rigor, this formalism is given, for instance, in [9, chapter VIII, sect. IV] and [7, § 33]; for mathematically rigorous exposition of the problem, see [10, chapter IV, sect. 8]. The formalism of density operator has also a bundle analogue which is described in the present section.

10.1. Hilbert space description (review)

Below we briefly recall the notions of a mixed state and density operator in the conventional Hilbert space description of quantum mechanics [9] (see also [20, 24]).

Consider a quantum system which at a moment t with a probability p_i can be found in a state with a state vector $\psi_i(t)\in\mathcal{F}$. Here i belongs to some set I of indices and the statistical weights p_i are assumed explicitly time-independent:

$$0\leq p_i\leq 1,\quad \sum_{i\in I}p_i=1,\quad \frac{\partial p_i}{\partial t}=0. \tag{10.1}$$

The state of such a system is described by the *density operator*

$$\rho(t):=\sum_{i\in I}\psi_i(t)\frac{p_i}{\langle\psi_i(t)|\psi_i(t)\rangle}\psi_i^\dagger(t)\colon\mathcal{F}\to\mathcal{F}. \tag{10.2}$$

Here with a dagger as a superscript are denoted the dual Hermitian conjugate vectors and spaces with respect to the inner product $\langle\cdot|\cdot\rangle$, i.e. if $\psi\in\mathcal{F}$, then $\psi^\dagger\in\mathcal{F}^\dagger$ is a map $\psi^\dagger\colon\mathcal{F}\to\mathbb{C}$ such that $\psi^\dagger\colon\chi\mapsto\langle\psi|\chi\rangle$ for $\chi\in\mathcal{F}$, and a product like $\psi\chi^\dagger$, $\psi,\chi\in\mathcal{F}$ is defined as an operator $\psi\chi^\dagger\colon\mathcal{F}\to\mathcal{F}$ via $(\psi\chi^\dagger)(\varphi):=(\chi^\dagger(\varphi))\psi=\langle\chi|\varphi\rangle\psi$ for $\varphi\in\mathcal{F}$.[52] The density operator (10.2)

[52] In Dirac's notation ψ, $\psi^\dagger$, and $\psi\chi^\dagger$ will look like $|\psi\rangle$, $\langle\psi|$ and $|\psi\rangle\langle\chi|$ respectively [7,9]. Notice that $(\psi\chi^\dagger)^\dagger=\chi\psi^\dagger$ corresponds to $(|\psi\rangle\langle\chi|)^\dagger=|\chi\rangle\langle\psi|$ in Dirac's notation.

is Hermitian, positive definite, trace-class, and of unit trace [9, 10]. So, we have

$$\rho^{\dagger}(t) = \rho(t), \quad \langle\psi(t)|\rho(t)\psi(t)\rangle \geq 0, \quad \operatorname{Tr}\rho(t) = 1, \tag{10.3}$$

where Tr denotes the trace of an operator. Conversely, every such operator is a density operator (in the absence of superselection rules) [10, chapter IV, subsect. 8.6].

By definition the *mean (expectation) value* of an observable $\mathcal{A}$ is

$$\langle\mathcal{A}\rangle_{\rho}^{t} := \langle\mathcal{A}(t)\rangle_{\rho}^{t} := \operatorname{Tr}(\rho(t)\circ\mathcal{A}(t)) \tag{10.4}$$

for a system whose state is described by a density operator $\rho(t)$. It is interpreted as physically observable value of a dynamical variable $\mathbb{A}$ represented by $\mathcal{A}$.

The time evolution of the density operator is described by postulating the Schrödinger equation for it, called also *von Neumann's or Liouville equation*:

$$\mathrm{i}\hbar\frac{d\rho(t)}{dt} = [\mathcal{H}(t), \rho(t)]_{-} := \mathcal{H}(t)\circ\rho(t) - \rho(t)\circ\mathcal{H}(t) \tag{10.5}$$

where $\mathcal{H}(t)$ is system's Hamiltonian. If $\rho(t_0)$ is known for some instant of time t_0, then

$$\rho(t) = \mathcal{U}(t, t_0)\circ\rho(t_0)\circ\mathcal{U}^{-1}(t, t_0), \tag{10.6}$$

where $\mathcal{U}(t, t_0)$ is the system's evolution operator (see Sect. 2). In fact, (10.6) is the general solution of (10.5) with respect to $\rho(t)$.[53]

If the sum in (10.2) contains only one term or terms which are proportional up to a phase factor to one of them, the system is said to be in a *pure state* (described by (one of) the corresponding state vector(s) entering in (10.2)); otherwise the system's state is called *mixed*. The equality $\rho^2(t) = \rho(t)$ is a criterion for a state to be pure [9, 10].

10.2. Hilbert bundle description

The fibre bundle approach to quantum mechanics developed until now concerns only pure states of the quantum systems. In this subsection we are going to show that it admits a natural modification which covers the bundle description of systems' mixed states. The idea for including mixed states in the new approach is quite simple: applying the 'principle of invariance of the observed values' (see the comments after postulates 7.1 in Subsect. 7.2) one should 'mend' the definition of a 'bundle mean value' in a way analogous to the transition from (2.11) to (10.4) and, besides this, on the base of (10.2) postulate 4.2 should appropriately be changed. A realization of such kind of procedure is proposed in the present subsection.

[53] Equation (10.5) is equivalent to the assumption that every vector $\psi_i(t)$ in (10.2) evolves according to the Schrödinger equation (2.6). Respectively, equation (10.6) is equivalent to (2.1) for the vectors $\psi_i(t)$.

10.2.1. Heuristic considerations

The easiest way to introduce the bundle analogue of the density operator $\rho(t)$ is via (10.4). Expressing $\mathcal{A}$ from (7.1) and substituting the result into (10.4), we find

$$\langle \mathcal{A}(t)\rangle_\rho^t = \langle A_\gamma(t)\rangle_{\mathrm{P}_\gamma}^t \tag{10.7}$$

where

$$\langle A_\gamma(t)\rangle_{\mathrm{P}_\gamma}^t = \mathrm{Tr}(\mathrm{P}_\gamma(t)\circ A_\gamma(t)) \tag{10.8}$$

is the (bundle) mean value of the morphism A, corresponding to $\mathcal{A}$, in a state characterized by the *density lifting of paths* $\mathrm{P}\colon \gamma \mapsto \mathrm{P}_\gamma\colon t \mapsto \mathrm{P}_\gamma(t)$ defined via (cf. (7.1))

$$\mathrm{P}_\gamma(t) := l_{\gamma(t)}^{-1}\circ\rho(t)\circ l_{\gamma(t)}\colon F_{\gamma(t)}\to F_{\gamma(t)}. \tag{10.9}$$

Meanwhile, equation (10.7) expresses the natural requirement that the expectation value of a dynamical variable $\mathbb{A}$ must be independent of the (mathematical) way we calculate it.

The density lifting has also a representation like (10.2). In fact, substituting (4.1) and its Hermitian conjugate, i.e.

$$\psi^\dagger(t) = \Psi_\gamma^\ddagger(t)\circ l_{\gamma(t)}, \tag{10.10}$$

where $\Psi_x^\ddagger\colon F_x\to\mathbb{C}$ is defined by $\Psi_x^\ddagger\colon \Phi_x\mapsto\langle\Psi_x|\Phi_x\rangle_x$ for $\Psi_x,\Phi_x\in F_x$, $x\in M$,[54] for the vectors $\psi_i(t)$ (appearing in (10.2)) into (10.2), we get

$$\rho(t) = l_{\gamma(t)}\circ\mathrm{P}_\gamma(t)\circ l_{\gamma(t)}^{-1} \tag{10.11}$$

which is equivalent to (10.9) with

$$\mathrm{P}_\gamma(t) = \sum_{i\in I}\Psi_{i,\gamma}(t)\frac{p_i}{\langle\Psi_{i,\gamma}(t)|\Psi_{i,\gamma}(t)\rangle}\Psi_{i,\gamma}^\ddagger(t). \tag{10.12}$$

Here a product like $\Phi_x\Psi_x^\ddagger$ is considered as an operator $\Phi_x\Psi_x^\ddagger\colon F_x\to F_x$ such that $(\Phi_x\Psi_x^\ddagger)\mathrm{X}_x := \langle\Psi_x|\mathrm{X}_x\rangle_x\Phi_x,\ \mathrm{X}_x\in F_x$.

The above results show that the transition from Hilbert space to Hilbert bundle description of mixed states is achieved by replacing the vectors in $\mathcal{F}$ (resp. the operators acting on $\mathcal{F}$) with liftings of paths according to the general rules of sections 4 and 7. As we shall see, this observation has a general validity.

[54] Equation (10.10) is a consequence of the unitarity of l_x (see (3.6)).

10.2.2. Rigorous considerations

Above we have made a number of implicit assumptions which should be clearly formulated as explicit assertions. As mentioned earlier, for this end some modifications of the basic postulates of the bundle approach to quantum mechanics of pure states is required. Postulate 4.1 introduces and describes the general mathematical arena on which the objects of bundle version of quantum mechanics play their roles. It is so formulated that a modification for the inclusion of mixed states in the scheme is not required. But the situation with postulate 4.2 is completely different: it concerns the concrete way by means of which the pure states are represented on the scene provided by postulate 4.1. Therefore this assertion needs a radical change for the inclusion of mixed states in its range of validity. In a 'middle' position is postulate 7.1: in it the only referring to description of pure states is via equation (7.2) defining the mean values. Hence it needs only a 'cosmetic' repairing.

Taking into account the material of Subsect. 10.2.1, the new forms of postulates 4.2 and 7.1 and definition 4.2, which describe mixed and pure states equally well, are the following.

Postulate 10.1. *Let $J \subseteq \mathbb{R}$ be real interval representing the period of time in which a quantum system is investigated, (F, π, M) be its Hilbert bundle, and $\gamma\colon J \to M$ be a C^1 path in the base M. In (F, π, M) the state of the system at a moment $t \in J$ is described by a map* P *assigning to the pair (γ, t) a mapping $\mathrm{P}_\gamma(t)\colon F_{\gamma(t)} \to F_{\gamma(t)}$ such that*

$$\mathrm{P}_\gamma(t) := l^{-1}_{\gamma(t)} \circ \rho(t) \circ l_{\gamma(t)} \tag{10.13}$$

where $\rho(t)\colon \mathcal{F} \to \mathcal{F}$ is the conventional density operator describing the system's state at a moment t in the ordinary quantum mechanics.

Postulate 10.2. *Let (F, π, M) be the Hilbert bundle of a quantum system, $\gamma\colon J \to M$, and $t \in J$. In the bundle description of quantum mechanics, every dynamical variable $\mathbb{A}$ characterizing the system is represented by a map A assigning to the pair (γ, t) a map $A_\gamma(t)\colon \pi^{-1}(\gamma(t)) \to \pi^{-1}(\gamma(t))$ such that*

$$A_\gamma(t) = l^{-1}_{\gamma(t)} \circ \mathcal{A}(t) \circ l_{\gamma(t)} \tag{10.14}$$

where $\mathcal{A}(t)\colon \mathcal{F} \to \mathcal{F}$ is the linear Hermitian operator (in the system's Hilbert space $\mathcal{F}$) representing $\mathbb{A}$ in the conventional quantum mechanics. If at a moment $t \in J$ the system is in a state characterized by a mapping $\mathrm{P}_\gamma(t)$, provided by postulate 10.1*, the observed value of $\mathbb{A}$ (or of A) with respect to γ at a moment t is equal to the mean value of $A_\gamma(t)$ in $\mathrm{P}_\gamma(t)$ which, by definition, is*

$$\langle A \rangle^{t,\gamma}_{\mathrm{P}} = \langle A_\gamma(t) \rangle^t_{\mathrm{P}_\gamma} = \mathrm{Tr}(\mathrm{P}_\gamma(t) \circ A_\gamma(t)) \tag{10.15}$$

Definition 10.1. The Hilbert bundle (resp. Hilbert space) description of the (quantum mechanics of) a quantum system is the description of its state and dynamical variables via the mappings P and A (resp. operators ρ and $\mathcal{A}$) in the Hilbert bundle (F, π, M) (resp. Hilbert space $\mathcal{F}$).

Notes 4.1–4.4 remain completely valid now, with respect to the case of mixed states.

The mapping P, introduced by postulate 10.1, can naturally be considered as a lifting of paths in the bundle $\mathrm{mor}_M(F, \pi, M)$ of restricted morphisms over M in the Hilbert bundle of states (F, π, M). Actually, defining P as $\mathrm{P}\colon \gamma \mapsto \mathrm{P}_\gamma$ with $\mathrm{P}_\gamma\colon t \mapsto \mathrm{P}(t)$, we get

$$\mathrm{P} \in \mathrm{PLift}(\mathrm{mor}_M(F, \pi, M)). \tag{10.16}$$

Evidently, the map P can be regarded also as a, generally, multiple-valued section along paths $\mathrm{P}\colon \gamma \mapsto {}_\gamma\mathrm{P}$ with ${}_\gamma\mathrm{P}\colon x \mapsto \{\mathrm{P}_\gamma(t) | \gamma(t) = x,\ t \in J\}$ for $x \in \gamma(J)$. This situation is practically the same as the one with the mapping (state lifting/section) Ψ considered in Subsect. 4.3 due to which we are not going to repeat here the corresponding discussion.

Definition 10.2. The unique lifting of paths P or (multiple-valued) section along paths corresponding to the density operator ρ from conventional quantum mechanics will be called density lifting (of paths) or density section (along paths)

For brevity, we call, by abuse of the language, a particular value of P, say $\mathrm{P}_\gamma(t)$, a bundle density operator (at a moment t along γ).

Analogously to (4.4), we have the bijective correspondences

$$\begin{aligned}\text{DENSITY OPERATOR} &\iff \text{DENSITY LIFTING OF PATHS}\\ &\iff \text{DENSITY SECTION ALONG PATHS.}\end{aligned} \tag{10.17}$$

What concerns the observables, the transition from the description of pure to mixed states is formally expressed through the replacement of (7.2) with (10.15) as a definition of the mean value. Since the explicit definition of the mean value was not used in Sect. (7) after postulate 7.1, all of this section after postulate 7.1 is *in extenso* completely valid when mixed states are investigated. Consequently, all of the results contained in the mentioned piece of text can and will be applied freely in what follows in the present section. In particular, we notice the the validity of (7.5) and the full preservation of definition 7.1 in the case of mixed states.

Substituting (10.13) into (10.15) and using the invariance of the trace of composition of maps under their cyclic permutations, we get (10.7), i.e.

$$\langle A\rangle_{\mathrm{P}}^{t,\gamma} = \langle \mathcal{A}\rangle_\rho. \tag{10.18}$$

This simple equality is a concrete expression of the 'principle of invariance of the mean values' (Subsect. 7.2) which, in the present context, means the coincidence of the mean values of a dynamical variable calculated in the Hilbert bundle and Hilbert space descriptions of quantum mechanics. Expressed in other words, we can assert the complete coincidence of the predictions of the both, Hilbert bundle and Hilbert space, versions of quantum mechanics in the case when mixed states are involved.

If we insert (10.2) into (10.13) and take into account (4.3), we obtain the representation (10.12) for the density lifting P. It could be interpreted as follows: if a quantum system with probability p_i has a state lifting Ψ_i, then this system can be described via a density lifting given by (10.12).

From equation (10.12) or from (10.3), (10.9) and (3.6) follows that the bundle density operator $\mathrm{P}_\gamma(t)$ in $F_{\gamma(t)}$ is Hermitian, positive definite, trace-class, and of trace one, i.e.

$$\mathrm{P}_\gamma^\ddagger(t) = \mathrm{P}_\gamma(t), \quad \langle \Psi_\gamma(t) | \mathrm{P}_\gamma(t) \Psi_\gamma(t) \rangle_{\gamma(t)} \geq 0, \quad \mathrm{Tr}\,(\mathrm{P}_\gamma(t)) = 1. \tag{10.19}$$

In this sense, the bundle density operators are completely analogous to the ordinary density operator but they act on the corresponding fibres over the reference path γ, not in the Hilbert space $\mathcal{F}$.

10.3. Time evolution and equations of motion

The time evolution of the density lifting of paths, i.e. the relation between $\mathrm{P}_\gamma(t)$ and $\mathrm{P}_\gamma(t_0)$ for every $t, t_0 \in J$, can be found as follows. Substituting (10.11) for $t = t_0$ into (10.6), then inserting the result into (10.9), and, at last, applying (5.10), we get

$$\mathrm{P}_\gamma(t) = U_\gamma(t, t_0) \circ \mathrm{P}_\gamma(t_0) \circ U_\gamma^{-1}(t, t_0), \qquad t, t_0 \in J, \tag{10.20}$$

where $U_\gamma(t, t_0)$ is the evolution transport from $F_{\gamma(t_0)}$ to $F_{\gamma(t)}$. Therefore the system's evolution transport U governs the time-evolution (along γ) of the mixed states via (10.20).

A simple verification proves that the linear map $\mathrm{P}_\gamma(t_0) \mapsto \mathrm{P}_\gamma(t)$, defined by (10.20), satisfies (3.19) and (3.20). So, freely speaking, we may say that this is a 'transport-like' map by means of which P_γ is 'transported' along γ. This is something like 'representation' of the evolution transport in the space of liftings of γ. The rigorous study of this problem reviles that the pointed map is a linear transport along γ in the bundle $\mathrm{mor}_M(F, \pi, M)$ of point-restricted morphisms of (F, π, M). Indeed, comparing (10.20) with (3.47) and using (5.12), we see that (10.20) can equivalently be rewritten as (cf. (8.20))

$$\mathrm{P}_\gamma(t) = {}^\circ U_\gamma(t)(\mathrm{P}_\gamma(t_0)) \in (\pi_0^M)^{-1}(\gamma(t)) \tag{10.21}$$

where ${}^\circ U$ is the transport along paths in $\mathrm{mor}_M(F, \pi, M)$ associated to U. Consequently, the density lifting (10.16) is ${}^\circ U$-transported along the paths

in M. Thus, with respect to the density liftings, the transport ${}^{\circ}U$ plays the same role as the evolution transport U with respect to the state liftings (in the case of pure states).

Definition 10.3. The linear transport ${}^{\circ}U$ along paths, associated to the evolution transport U, will be called associated evolution transport.

Earlier we met the associated evolution transport in Subsect 8.2.3 during the exploration of the bundle Heisenberg picture of motion: we proved that it maps the Schrödinger observables to the Heisenberg ones via (8.21). It was established in Subsect. 8.2.3 that ${}^{\circ}U$ (by means of the generated by if lifting $\overline{{}^{\circ}U}$ of paths) satisfies the invariant equation (8.32) and is a solution of the initial-value problem (8.29) in which ${}^{\circ}D$ is the generate by ${}^{\circ}U$ derivation along paths. The associated evolution transport also appeared in Subsect 9.2 where we proved that it governs the time evolution of the integrals of motion via (9.12).

In conformity with definition 3.5, the above means that the density lifting P is a lifting generated by the associate evolution transport ${}^{\circ}U$ and, by virtue of (3.40), satisfies the equation

$$ {}^{\circ}D(\mathrm{P}) = 0 \tag{10.22}$$

which replace the bundle Schrödinger equation (6.27) for the states liftings in the general case of mixed states.[55] We call (10.22) *bundle Schrödinger equation for the density lifting* P. This equation is covariant in a sense that it is completely independent of any coordinates or observers (reference paths).

Now the time evolution of a quantum system described by a density lifting P can be formulated entirely in bundle terms. Let the system is characterized by the derivation ${}^{\circ}D$ along paths in $\mathrm{mor}_M(F,\pi,M)$.[56] Suppose the system's density lifting P^0_γ is known along some path $\gamma\colon J \to M$ at the point $t_0 \in J$. Then the density lifting P is the unique solution of (10.22) under the initial condition

$$ \mathrm{P}_\gamma(t_0) = \mathrm{P}^0_\gamma. \tag{10.23}$$

Due to (10.21), the explicit form of P is given via

$$ \mathrm{P}_\gamma(t) = {}^{\circ}U_\gamma(t,t_0)(\mathrm{P}^0_\gamma) \tag{10.24}$$

[55] Equation (10.22) is equivalent to the assumption that the liftings Ψ_i in (10.12) evolve according to the bundle Schrödinger equation (6.27).

[56] Equivalently, the system can be characterized via the bundle Hamiltonian H, the evolution transport U or the generate by it (resp. associated to it) derivation D (resp. transport ${}^{\circ}U$) along paths. Anyone of these quantities uniquely determines (and is determined by) the derivation ${}^{\circ}D$ generated by the associated evolution transport ${}^{\circ}U$.

where the associated evolution transport $^{\circ}U$ can be found as, e.g., the unique solution of the initial-value problem (8.29). Notice, this situation is similar to the one described in the beginning of Subsect. 6.2 when we studied bundle equations of motion for pure states.

An equivalent to (10.22) differential equation, corresponding to the evolution law (10.20), can be derived in the following way. Differentiating the matrix form of (10.20) with respect to t and applying equation (6.15), we obtain the *matrix-bundle Schrödinger equation for the density lifting* as

$$\mathrm{i}\hbar\frac{d\mathbf{P}_\gamma(t)}{dt} = [\boldsymbol{H}^{\mathbf{m}}_\gamma(t), \mathbf{P}_\gamma(t)]_- \tag{10.25}$$

with $\boldsymbol{H}^{\mathbf{m}}_\gamma(t)$ being the matrix-bundle Hamiltonian (given by (6.13)). This equation is the matrix-bundle analogue of (10.5), to which it is equivalent as it can be proved via the substitution of the matrix form of (10.11) into the one of (10.5) (see also (7.14)). Consequently, the results of Sect. 6 show that (10.20) gives the general solution (10.25) with respect to $\mathrm{P}_\gamma(t)$.

The matrix equation (10.25) can be written in an invariant form too. To this end we shall use the following result which is a simple corollary of (10.25), (6.20), and (6.22): If Ψ is a lifting of paths in (F, π, M) and one of the equations $D^\gamma_t(\Psi) = 0$, $D^\gamma_t[\mathrm{P}(\Psi)] = 0$, or (10.25) is valid, then the remaining two of them are equivalent.[57] From here follows that (10.25) is equivalent to the system

$$(D^\gamma_t \circ \mathrm{P})\Psi = 0, \tag{10.26a}$$

$$D^\gamma_t(\Psi) = 0. \tag{10.26b}$$

If we denote by $\tilde{\mathrm{P}}_\gamma(t)$ the restriction of $\mathrm{P}_\gamma(t)\colon F_{\gamma(t)} \to F_{\gamma(t)}$ on the set of (state) liftings of γ which are (linearly) transported along γ by means of the evolution transport, i.e. the ones satisfying (10.26b), then (10.26) is equivalent to

$$\tilde{D}(\tilde{\mathrm{P}}) = 0 \tag{10.27}$$

where $\tilde{D}$ is the differentiation along paths of bundle $\mathrm{mor}_M(F, \pi, M)$ of point-restricted morphisms (see equation (3.36)). The above discussion shows the equivalence of (10.27) and (10.25), a fact which is also an evident corollary of (6.33) and (6.22). The equation (10.27) is a version of the bundle *Schrödinger equation* (10.22) *for the density liftings*.

10.4. Representations in the different pictures of motion

Let us turn now our attention to the description of mixed states in the different pictures of motion (see Sect. 8).

[57] The action of $\mathrm{P} \in \mathrm{PLift}(\mathrm{mor}_M(F, \pi, M))$ on $\Psi \in \mathrm{PLift}(F, \pi, M)$ is defined by (3.34).

In the Schrödinger picture of the Hilbert bundle (resp. space) description of quantum mechanics, which, in fact, was investigated until now in this section, the motion of a quantum system is described by pairs like $(\mathrm{P}_\gamma(t), A_\gamma(t))$ (resp. $(\rho_\gamma(t), \mathcal{A}_\gamma(t))$) of generally time-depending mappings representing the density lifting (resp. operator) and some dynamical variable $\mathbb{A}$.

The transition to the *Heisenberg picture* is achieved via the general formulae (8.21) for the observable liftings and (8.9) for the operators in $\mathcal{F}$. In particular, for the density lifting P and operator ρ they, respectively, give:

$$\begin{aligned}\mathrm{P}^{\mathrm{H}}_{\gamma,t}(t_0) &:= U^{-1}_\gamma(t,t_0)\circ \mathrm{P}_\gamma(t)\circ U_\gamma(t,t_0) = {}^\circ U_\gamma(t_0,t)(\mathrm{P}_\gamma(t)) \qquad (10.28)\\ &= \mathrm{P}_\gamma(t_0)\colon F_{\gamma(t_0)} \to F_{\gamma(t_0)},\\ \rho^{\mathrm{H}}_t(t_0) &:= \mathcal{U}^{-1}(t,t_0)\circ\rho(t)\circ\mathcal{U}(t,t_0) = \rho_\gamma(t_0)\colon \mathcal{F}\to\mathcal{F} \qquad (10.29)\end{aligned}$$

where (10.20) and (10.6) were used. Consequently in the Heisenberg picture the Hilbert bundle and Hilbert space descriptions are by means of pairs like $\left(\mathrm{P}^{\mathrm{H}}_{\gamma,t}(t_0), A^{\mathrm{H}}_{\gamma,t}(t_0)\right) = \left(\mathrm{P}_\gamma(t_0), A^{\mathrm{H}}_{\gamma,t}(t_0)\right)$ and $\left(\rho^{\mathrm{H}}_t(t_0), \mathcal{A}^{\mathrm{H}}_t(t_0)\right) = \left(\rho(t_0), \mathcal{A}^{\mathrm{H}}_t(t_0)\right)$, respectively, in which the time dependence is entirely shifted from the density liftings and operators to the observables. So, in this picture the density liftings and operators are replaced by constant in time mappings, while the observables' evolution is governed by the Heisenberg equation of motion for them (see (8.12), or (8.10), or (8.30)).

Of course, in the Heisenberg picture the mean values remain unchanged:

$$\langle A^{\mathrm{H}}_{\gamma,t}(t_0)\rangle^{t_0}_{\mathrm{P}^{\mathrm{H}}_{\gamma,t}} = \langle \mathcal{A}^{\mathrm{H}}_t(t_0)\rangle^{t_0}_{\rho^{\mathrm{H}}_t} = \langle A_\gamma(t)\rangle^t_{\mathrm{P}_\gamma} = \langle \mathcal{A}(t)\rangle^t_\rho, \qquad (10.30)$$

where

$$\langle A^{\mathrm{H}}_{\gamma,t}(t_0)\rangle^{t_0}_{\mathrm{P}^{\mathrm{H}}_{\gamma,t}} := \mathrm{Tr}\left(\mathrm{P}_\gamma(t_0)\circ A^{\mathrm{H}}_{\gamma,t}(t_0)\right), \quad \langle \mathcal{A}^{\mathrm{H}}_t(t_0)\rangle^{t_0}_{\rho^{\mathrm{H}}_t} := \mathrm{Tr}\left(\rho(t_0)\circ \mathcal{A}^{\mathrm{H}}_t(t_0)\right). \qquad (10.31)$$

The chain equation (10.30) is a corollary of the invariance of the trace of a product (composition) of operators with respect to a cyclic permutation of the multipliers.

The shift from the Schrödinger to 'general' picture can be achieved by means of the general equations (8.38) and (8.41). Hence, in the V-picture of motion the density lifting P and operator ρ are replaced, respectively, by

$$\begin{aligned}\mathrm{P}^V_{\gamma,t}(t_1) &= V_\gamma(t_1,t)\circ\mathrm{P}_\gamma(t)\circ V^{-1}_\gamma(t_1,t)\colon F_{\gamma(t_1)}\to F_{\gamma(t_1)}, \qquad (10.32)\\ \rho^{\mathcal{V}}_t(t_1) &= \mathcal{V}_\gamma(t_1,t)\circ\rho_\gamma(t)\circ\mathcal{V}^{-1}(t_1,t)\colon \mathcal{F}\to\mathcal{F}. \qquad (10.33)\end{aligned}$$

Since in the V-picture the Hilbert bundle (resp. space) description is via pairs like $\left(\mathrm{P}^V_{\gamma,t}(t_1), A^V_{\gamma,t}(t_1)\right)$ $\left(\text{resp. } \left(\rho^{\mathcal{V}}_t(t_1), \mathcal{A}^{\mathcal{V}}_t(t_1)\right)\right)$, the mean values of the observables remain unchanged, as in the Heisenberg picture:

$$\langle A^V_{\gamma,t}(t_1)\rangle^{t_1}_{\mathrm{P}^V_{\gamma,t}} = \langle \mathcal{A}^{\mathcal{V}}_t(t_1)\rangle^{t_1}_{\rho^{\mathcal{V}}_t} = \langle A_\gamma(t)\rangle^t_{\mathrm{P}_\gamma} = \langle \mathcal{A}(t)\rangle^t_\rho, \qquad (10.34)$$

where

$$\begin{aligned}\langle A^V_{\gamma,t}(t_1)\rangle^{t_1}_{\mathrm{P}^V_{\gamma,t}} &:= \mathrm{Tr}\left(\mathrm{P}^V_{\gamma,t}(t_1)\circ A^V_{\gamma,t}(t_1)\right),\\ \langle \mathcal{A}^{\mathcal{V}}_t(t_1)\rangle^{t_1}_{\rho^{\mathcal{V}}_t} &:= \mathrm{Tr}\left(\rho^{\mathcal{V}}_t(t_1)\circ \mathcal{A}^{\mathcal{V}}_t(t_1)\right).\end{aligned} \tag{10.35}$$

In the V-picture the density liftings, operators, and observables generally change in time. For all of them this evolution is governed by the equations (8.52) and (8.50), but for the density liftings and operators they can be written in a more concrete form. For this purpose, we have to calculate the last terms in the r.h.s. of (8.52) and (8.50).

Using (8.38) and (10.5), we obtain

$$\left(\frac{\partial\rho(t)}{\partial t}\right)^{\mathcal{V}}_t(t_1) = \mathcal{V}(t_1,t)\circ\frac{\partial\rho(t)}{\partial t}\circ\mathcal{V}^{-1}(t_1,t) = \frac{1}{\mathrm{i}\hbar}\left[\mathcal{H}^{\mathcal{V}}_t(t_1),\rho^{\mathcal{V}}_t(t_1)\right]_-.$$

Analogously, applying (8.44) and the just get equation, we find

$$\left(\frac{\partial\rho(t)}{\partial t}\right)^{V}_{\gamma,t}(t_1) = l^{-1}_{\gamma(t_1)}\circ\left(\frac{\partial\rho(t)}{\partial t}\right)^{\mathcal{V}}_t(t_1)\circ l_{\gamma(t_1)} = \frac{1}{\mathrm{i}\hbar}\left[H^V_{\gamma,t}(t_1),\mathrm{P}^V_{\gamma,t}(t_1)\right]_-.$$

At last, substituting the above two equations into (8.52) and (8.50), we, respectively, get

$$\mathrm{i}\hbar\frac{\partial\mathrm{P}^V_{\gamma,t}(t_1)}{\partial t} = \left[\widetilde{H}^V_{\gamma,t}(t_1),\mathrm{P}^V_{\gamma,t}(t_1)\right]_-, \tag{10.36}$$

$$\mathrm{i}\hbar\frac{\partial\rho^{\mathcal{V}}_t(t_1)}{\partial t} = \left[\widetilde{\mathcal{H}}^{\mathcal{V}}_t(t_1),\rho^{\mathcal{V}}_t(t_1)\right]_- \tag{10.37}$$

where (8.48) and (8.53) were taken into account. These are the equations of motion for the density lifting and operator in the V-picture.

If the evolution transport and operator are known (in the V-picture), then combining, from one hand, (10.32), (10.20) and (8.59) and, from other hand, (10.33), (10.6), and (8.58), we get the general solution of (10.36) and (10.37), respectively, in the form

$$\mathrm{P}^V_{\gamma,t}(t_1) = U^V_\gamma(t,t_1,t_0)\circ\mathrm{P}_{\gamma,t_0}(t_1)\circ\left(U^V_\gamma(t,t_1,t_0)\right)^{-1}, \tag{10.38}$$

$$\rho^{\mathcal{V}}_t(t_1) = \mathcal{U}^{\mathcal{V}}(t,t_1,t_0)\circ\rho_{t_0}(t_1)\circ\left(\mathcal{U}^{\mathcal{V}}(t,t_1,t_0)\right)^{-1}. \tag{10.39}$$

As one can expect, in the case of Heisenberg picture, due to (8.60), these formulae reproduce (10.28) and (10.29) respectively.

11. Curvature of the evolution transport

The curvature of a linear transport along paths could be considered as a measure of its dependence on the transportation path [66,67]; in particular,

the transportation of a vector by means of a curvature-free transport depends only on the initial and final points of the transportation (in the base) but not on the concrete path along which it is performed. These results are important for us in connection with the interpretation of the reference path $\gamma\colon J \to M$ as a trajectory (or world line) of some observer. In this sense the curvature of the evolution transport is a measure of its observer-dependence and, consequently, it reflects the observer-dependence of the quantum evolution as a whole. In particular, the evolution of a system described via curvature-free evolution transport has an absolute character in a sense that it is independent of the reference path (observer) with respect to which it is investigated. By these reasons, below we shall find a necessary and sufficient condition for the evolution transport to be with zero curvature.

At first, we define the curvature of the evolution transport (cf. [66, sect. 3]).

Let $\mathsf{D}\colon \gamma \mapsto \mathsf{D}^\gamma$, $\gamma\colon J \to M$ be the section-derivation along paths induced by the derivation D along paths generated by the evolution transport U, i.e. (see (3.31), (3.32) and the comments after them)

$$\mathsf{D}^\gamma\colon\ \mathrm{Sec}^1(F,\pi,M)|_{\gamma(J)} \to \mathrm{Sec}^0(F,\pi,M)|_{\gamma(J)} \tag{11.1}$$

and if σ is a C^1 section (defined in a (neighborhood of) $\gamma(J)$) and $x \in \gamma(J)$, we have

$$(\mathsf{D}^\gamma\sigma)(x) := D^\gamma_s\hat{\sigma}, \qquad \gamma(s) = x,\ s \in J,\ \hat{\sigma} := \sigma \circ \gamma \tag{11.2}$$

for injective path γ and $(\mathsf{D}^\gamma\sigma)(x) := \{D^\gamma_s\hat{\sigma} | \gamma(s) = x,\ s \in J\}$ if γ has self-intersection points.

Let J and J' be real intervals and the mapping $\eta\colon J \times J' \to M$ be such that the paths $\eta(\cdot,t)\colon s \mapsto \eta(s,t)$ and $\eta(s,\cdot)\colon t \mapsto \eta(s,t)$ for $(s,t) \in J \times J'$ be injective, i.e. without self-intersections, and $\eta(s,J') \cap \eta(s',J') = \eta(J,t) \cap \eta(J,t') = \varnothing$ for $(s,t),(s',t') \in J \times J'$, $s \neq s'$ and $t \neq t'$; so the paths $\eta(\cdot,t)$, $t \in J'$ or $\eta(s,\cdot)$, $s \in J$ do not intersect each other.

The curvature (operator) R of the evolution transport U is a mapping $R\colon \eta \mapsto R^\eta$ with $R^\eta\colon (s,t) \mapsto R^\eta(s,t)$, $(s,t) \in J \times J'$ where the mapping

$$R^\eta(s,t)\colon\ \mathrm{Sec}^2(F,\pi,M)|_{\eta(J,J')} \to \pi^{-1}(\eta(s,t)) \tag{11.3}$$

is defined as follows. For every $\sigma \in \mathrm{Sec}^2(F,\pi,M)|_{\eta(J,J')}$, we define the sections $\sigma_1, \sigma_2 \in \mathrm{Sec}^1(F,\pi,M)|_{\eta(J,J')}$ by

$$\sigma_1(\eta(s,t)) := \big(\mathsf{D}^{\eta(s,\cdot)}(\sigma|_{\eta(s,J')})\big)(\eta(s,t)) = D_t^{\eta(s,\cdot)}(\hat{\sigma}) \in \pi^{-1}(\eta(s,t)),$$
$$\sigma_2(\eta(s,t)) := \big(\mathsf{D}^{\eta(\cdot,t)}(\sigma|_{\eta(J,t)})\big)(\eta(s,t)) = D_s^{\eta(\cdot,t)}(\hat{\sigma}) \in \pi^{-1}(\eta(s,t)),$$

where $\hat{\sigma} \in \mathrm{PLift}(F,\pi,M)|_{\eta(J,J')}$ and $\hat{\sigma}\colon \gamma \mapsto \hat{\sigma}_\gamma := \sigma \circ \gamma$ for every path γ in $\eta(J,J')$.

The action of the mapping (11.3) on σ is

$$(R^\eta(s,t))\sigma := \big(\mathsf{D}^{\eta(\cdot,t)}(\sigma_1|_{\eta(J,t)})\big)(\eta(s,t)) - \big(\mathsf{D}^{\eta(s,\cdot)}(\sigma_2|_{\eta(s,J')})\big)(\eta(s,t)) \\ = D_s^{\eta(\cdot,t)}(\hat\sigma_1) - D_t^{\eta(s,\cdot)}(\hat\sigma_2) \quad (11.4)$$

where $\hat\sigma_1, \hat\sigma_2 \in \mathrm{PLift}(F,\pi,M)|_{\eta(J,J')}$ and $\hat\sigma_\alpha \colon \gamma \mapsto (\hat\sigma_\alpha)_\gamma := \sigma_\alpha \circ \gamma$, $\alpha = 1,2$, for every path γ in $\eta(J,J')$.

Symbolically, by abuse of the notation, one may write

$$R^\eta(s,t) := \mathsf{D}^{\eta(\cdot,t)} \circ \mathsf{D}^{\eta(s,\cdot)} - \mathsf{D}^{\eta(s,\cdot)} \circ \mathsf{D}^{\eta(\cdot,t)} \quad (11.5)$$

but, as a consequence of (11.1), compositions like $\mathsf{D}^{\eta(\cdot,t)} \circ \mathsf{D}^{\eta(s,\cdot)}$ are not quite correctly defined since the action of this expression on σ must be defined as $\mathsf{D}^{\eta(\cdot,t)}(\sigma_1|_{\eta(J,t)})$.

If $\{e_a(x)\}$, $x \in \eta(J,J')$, is a basis in $\pi^{-1}(x) = F_x$, the local expansion (6.20) can be apply for explicit calculation of the derivatives $D_t^{\eta(s,\cdot)}\hat\sigma$, $D_s^{\eta(\cdot,t)}\hat\sigma$, $D_s^{\eta(\cdot,t)}\hat\sigma_1$, $D_t^{\eta(s,\cdot)}\hat\sigma_2$. The substitution of these expansions into (11.4) results in the assertion that the mapping (11.3) is linear, that is

$$(R^\eta(s,t))(\sigma) = (R^\eta(s,t))^a{}_b\, \sigma^b(\eta(s,t))e_a(\eta(s,t)), \quad (11.6)$$

and its local components in $\{e_a\}$ are

$$(R^\eta(s,t))^a{}_b = \frac{\partial}{\partial s}\left[\Gamma^a{}_b(t;\eta(s,\cdot))\right] - \frac{\partial}{\partial t}\left[\Gamma^a{}_b(s;\eta(\cdot,t))\right] \\ + \Gamma^a{}_c(s;\eta(\cdot,t))\Gamma^c{}_b(t;\eta(s,\cdot)) - \Gamma^a{}_c(t;\eta(s,\cdot))\Gamma^c{}_b(s;\eta(\cdot,t)). \quad (11.7)$$

Here $\Gamma^{\cdots}_{\cdots}(\cdots)$ are the corresponding coefficients of the evolution transport in $\{e_i(x)\}$ defined by (6.21). Using the fundamental relation (6.22), we can explicitly calculate the curvature. The easiest way to do this is to choose the bases $\{e_a(x)\}$ and $\{f_a(t)\}$ such that $\boldsymbol{l}_x(t) = [\delta^a_b] = \mathbb{1}$ (see remark 6.1). Then $\boldsymbol{E}(t) = 0$ and $\boldsymbol{H}^{\mathbf{m}}_\gamma(t) = \boldsymbol{H}(t) = \boldsymbol{\mathcal{H}}(t)$, $\gamma = \eta(\cdot,t), \eta(s,\cdot)$. Hence, due to (6.22), now equation (11.7) reduces to

$$\boldsymbol{R}^\eta(s,t) = \frac{1}{(-\mathrm{i}\hbar)^2}[\boldsymbol{\mathcal{H}}(s), \boldsymbol{\mathcal{H}}(t)]_- \quad (11.8)$$

where we have assumed, as usual, that the conventional Hamiltonian $\mathcal{H}(t)$ is independent of the observer's trajectory (reference path) γ.[58] (The last equality is valid only in the special bases in which it is derived!)

From (11.8) we infer that the *evolution transport is curvature free if and only if the values of the Hamiltonian at different moments commute,* viz.

$$R^\eta = 0 \iff [\mathcal{H}(s), \mathcal{H}(t)]_- = 0. \quad (11.9)$$

[58] If $\mathcal{H}$ depends explicitly on the reference path γ, $\mathcal{H} = \mathcal{H}(t;\gamma)$, in the r.h.s. of (11.8) the term $\left[\frac{\partial}{\partial s}\boldsymbol{\mathcal{H}}(t;\eta(s,\cdot)) - \frac{\partial}{\partial t}\boldsymbol{\mathcal{H}}(s;\eta(\cdot,t))\right]/(-\mathrm{i}\hbar)^2$ will appear.

In particular, this is true for explicitly time-independent Hamiltonians, i.e. for ones with $\partial\mathcal{H}(t)/\partial t = 0$. According to (7.27), the bundle form of (11.9) is

$$R^{\eta} = 0 \iff \left[\breve{H}_{\gamma,s}(r), \breve{H}_{\gamma,t}(r)\right]_{-} = 0. \tag{11.10}$$

for some (and hence any) path $\gamma\colon J \to M$. Here $r, s, t \in J$ and $\breve{H}_{\gamma,t}(r)$ is (the transported by means of $l^{\gamma}_{s\to t}$ Hamiltonian which is) calculated via (7.24).

Consider now a curvature free evolution transport on $W = \eta(J, J')$, i.e. $R^{\eta}(s,t) \equiv 0$ for every $(s,t) \in J \times J'$. From [66, proposition 3.3] we know that in this case there exists a field of base $\{e_i\}$ over W in which the transport's coefficients vanish along any path γ in W. In it, due to (6.15) and (6.22), we have

$$\widetilde{\boldsymbol{U}_{\gamma}}(t,t_0) = \mathbb{1}, \quad \widetilde{\boldsymbol{\Gamma}}_{\gamma}(t) = \boldsymbol{0}, \quad \widetilde{\boldsymbol{H}^{\mathrm{m}}_{\gamma}}(t) = \boldsymbol{0} \qquad \text{for every } \gamma. \tag{11.11}$$

Notice, equations (8.5) are valid for *arbitrary* evolution transports along any *fixed* path in appropriate bases along it, while equations (11.11) hold only for *curvature free* evolution transports in a suitably chosen field of bases in a *whole* set W. The connection of the special bases in which (11.11) is true with the Heisenberg picture is the same as discussed in Subsect 8.2 for the bases in which (8.5) are satisfied.

We want to emphasize on the fact that according to (6.13) the local vanishment of the matrix-bundle Hamiltonian does not imply the same property for the Hamiltonian as an operator or lifting.

According to [67, theorem 6.1] a linear transport along paths is path-independent iff it is flat, i.e. with vanishing curvature, provided the corresponding paths lie entirely in the region of flatness of the transport. Applying this result to the evolution transport U, we can assert that only the curvature free evolution transports are path independent. Physically, due to (5.7) and/or (10.20), this means that the evolution of quantum systems with curvature free evolution transports is independent of the reference path γ with respect to which it is described. Since we interpret γ as observer's trajectory (world line),[59] the last result has a meaning of observer-independence of the time-evolution of such systems. As we proved above, for a sufficient general class of quantum systems the evolution transport is flat (=curvature free) iff equation (11.9) holds, i.e. if the (ordinary Hilbert space) Hamiltonian of a system is such that its different time values commute. An evident sub-class of quantum systems with flat evolution transports is the one of systems described via explicitly time-independent Hamiltonians. This sub-class is also broad enough and covers a great number of physically important cases [7–9].

[59] Physically we interpret the paths $\eta(s,\cdot)\colon s \mapsto \eta(s,t)$ and $\eta(\cdot,t)\colon s \mapsto \eta(s,t)$ as trajectories (world lines) of observers with (proper) times t and s respectively.

12. On observer's rôle and theory's interpretation

The concept of an 'observer' is more physical than mathematical one as it is not very well mathematically rigorously defined; sometimes its meaning is more intuitive than strict one. Generally an '*observer*' is a physical system whose state is assumed to be 'completely' known and which has a double rôle with respect to the other systems(s) which is (are) under consideration. From one hand, it provides certain reference point, generally a set of objects and their properties, with respect to which are determined (all of) the quantities characterizing the investigated system(s) in some problem. From the other hand, it is supposed the 'observer' can perform certain procedures, called 'measurements' or 'observations', by means of which 'he' finds (determines) the parameters, properties, quantities, etc. describing the studied system(s). This second rôle of the observers is out of the subject of the present investigation and will not be discussed here (for some its aspects see, e.g., [9, 10]).

In this work the observers are supposed to be local and point-like, i.e. they are material points that can perform measurements at the points at which they are situated. They are moving (along paths) in some differentiable manifold M.[60] Namely their trajectories (world lines in special or general relativity interpretation (see below)) are the reference objects with respect to which we study the behaviour of the quantum systems. The 'observational' properties of an observer are assumed fixed and such that: (i) Allow the observer to determine the initial values of the quantities characterizing the state of the studied system(s) at some instant of time t_0; (ii) Give certain correspondence rules according to which to any dynamical variable $\mathbf{A}$, connected to the investigated system(s), is assigned some observable which is a Hermitian operator (resp. lifting of paths), say $\mathcal{A}$ (resp. A), in the Hilbert space (resp. Hilbert bundle) description that has a complete set of (maybe orthonormal) eigenvectors (resp. eigen-liftings of paths).

All quantities in the present work are referred to an observer moving along some path $\gamma\colon J \to M$ parameterized with $t \in J$, where $J \subseteq \mathbb{R}$ is a real interval. Our intention is to interpret t as a 'time'. The possibility for such an interpretation is connected with the specific choice of the manifold M.

If we assume M to be the classical, coordinate, 3-dimensional Euclidean space $\mathbb{E}^3$ of the classical and quantum mechanics, which we have done more implicitly than explicitly throughout this investigation, there exists a *global time* $t_{\mathrm{g}} \in (-\infty, +\infty) = \mathbb{R}$ (in the Newtonian sense). In this case the trajectories, such as γ, of *all* observers is natural and convenient to be parameterized with this global time which, in fact, is done in the conventional, Hilbert space, quantum mechanics. (One can also parameterize each ob-

[60] Mathematically the developed here theory is sensible also if M is considered as a 'more general' object than a manifold, but, at present, there are not indications that such a theory can be physically important.

server's trajectory with its own local time which is in bijective, usually C^1, correspondence with t_{g}, but this is an inessential generalization as t_{g} itself is defined with some arbitrariness (usually a C^1 map $\mathbb{R} \to \mathbb{R}$).) Consequently the assumptions $M = \mathbb{E}^3$, $J = \mathbb{R}$ and $t = t_{\mathrm{g}}$ are necessary and sufficient for the full equivalence of the Hilbert space and Hilbert bundle descriptions of quantum mechanics. These assumptions lead to one inessential mathematical complication: since the observer's trajectory γ can has self-intersections, the sections along γ and morphisms along γ can be multiple-valued at the points of self-intersection of γ, if any. But this does not have some serious consequences, especially if one works with liftings of paths which are always single-valued.

Analogous to $M = \mathbb{E}^3$ is the case when the Hilbert bundle is identified with the system's configuration space. The only difference is that $\gamma\colon J \to M$ has to be interpreted as the trajectory of the system in this space rather then the one of some observer. Practically the same is the case when as M is taken the system's phase space but, since this situation has some peculiarities, we shall comment on it in Sect. 14.

A similar, but slightly different, is the situation when M is taken to be the four dimensional Minkowski space-time M^4 of special relativity.[61] Nevertheless that now every observer has its own local preferred time, called proper or eigen-time, there is a global (e.g. coordinate) time t_{g} which is in bijective correspondence with these local times. In this case the observers are moving along paths, such as γ, which are their *world lines (paths or curves)* and usually are parameterized with the global time t_{g}. This is an important moment because the world lines of the real objects observed until now can not have self-intersections. Therefore, since the observers are supposed to be such, their world lines are without self-intersections. This implies the absence of the complication mentioned for $M = \mathbb{E}^3$, viz. now the sections along γ and morphisms along γ are single-valued and, in fact, are respectively sections and morphisms of the restriction $(F, \pi, M)|_{\gamma(J)}$ of the Hilbert bundle (F, π, M) on the set $\gamma(J)$. Otherwise the case $M = M^4$ is identical with the one with $M = \mathbb{E}^3$. Therefore, we can say that it represents the Hilbert bundle description of the nonrelativistic quantum mechanics over the special relativity space-time. This point is worth-mentioning as at it meet the relativistic and non-relativistic concepts whose unification leads to the relativistic quantum theory which will be considered elsewhere.

Another important possibility is $M = V_4$ where V_4 is the four dimensional pseudo-Riemannian space-time of general relativity.[62] The crucial

[61] A like construction, under the name 'Schrödinger bundle', is introduced in the paragraph containing equation (4) of [68]. This is a Hilbert bundle over M^4 having as a (standard) fibre $\mathcal{F} \times \mathcal{F}^*$ instead of the conventional Hilbert space $\mathcal{F}$ in our case.

[62] Almost the same is the case when M represents the space-time model of other gravitational theories, like Einstein-Cartan and the metric-affine ones, in which M is a curved (non-flat) manifold with respect to some linear connection.

point here is the generic non-existence of some global time, so the *world line* of any particular observer, say γ, with necessity has to be parameterize with its (specifically local) *proper time* t. A consequence of this is that in the (conventional) Hilbert space description the parameter ('time') t also has to be considered as a local (proper) time for the observer which describes the quantum system under consideration. Hence the global sections and morphisms defined via (4.2) and (7.9) have no physical meaning now. As in special relativity case, $M = M^4$, now γ cannot has self-intersections; so the sections and morphisms along γ are single-valued. What concerns other aspects of the case $M = V_4$, it is identical with the one for $M = \mathbb{E}^3$. Consequently, it represents the non-relativistic quantum mechanics over the general relativity space-time. Here we see, again, a meeting of (general) relativistic and nonrelativistic concepts whose unification will be investigated elsewhere.

Now we want to call attention to a particular 'degenerate' case which falls out of our general interpretation of $\gamma : J \to M$ as an observer's world line (trajectory). Namely, it is possible to put $M = J$, J being a real interval. For instance, as we pointed at the end of Subsect. 6.2, the considerations of [35] correspond to the choice $M = \mathbb{R}_+ = \{t : t \in \mathbb{R},\ t \geq 0\}$. Thus, if $M = J$, then $\gamma : J \to J$ and, as we tend to interpret $t \in J$ as time, it is natural now to assume that γ is bijective smooth (C^1) map.[63] If so, $\gamma(t) \in J$ is also a 'presentation' of the time, but in other (re)parameterization. In this case γ is without self-intersections and, consequently, the sections and morphisms along γ are simply (single-valued) sections and morphisms of the bundle (F, π, J) over the one-dimensional base J. Therefore, in this situation, we can say that the evolution of a quantum system is described via linear transportation (of the state) sections of (F, π, J) along the time, respectively the evolution of the observables is represented by linear transportation of (the observable) liftings/morphisms of (F, π, J) along the time. Let us note that now the connection with the observers is not completely lost as the time does not exist by 'itself' in the theory; it is connected with (measured by) a concrete observer regardless of the fact that it can be global of local (see above).

Another degenerate case is when M consist of a single point, $M = \{x\}$. Then $F = \pi^{-1}(x) =: F_x$ and $\gamma\colon J \to \{x\}$. Therefore $\gamma(t) \equiv x$ for every $t \in J$ and, if J is not compressed into a single real number, γ self-intersects at x infinitely many times. Also we have $l_{\gamma(t)} = l_{\gamma(s)} = l_x =: l$, $s, t \in J$ with $l\colon F \to \mathcal{F}$ being an isomorphism. So, in this way (see note 4.4), we obtain an isomorphic image in F of the quantum mechanics in $\mathcal{F}$. Evidently, the conventional quantum mechanics is recovered by the choices $\mathcal{F} = F$ and $l = \mathsf{id}_F$. Of course, now we can not interpreted γ as observer's trajectory or world line but the interpretation of t as a 'time' can be preserved.

[63] The case considered in reference [35] corresponds to $\gamma = \mathsf{id}_{\mathbb{R}_+}$.

If the quantum system under consideration has a classical analogue, then the manifold M can be identified with the system's configuration or phase space. In this case the path $\gamma\colon J \to M$ can be taken to be the trajectory of the system's classical analogue in the corresponding space. Thus we obtain an interesting situation: the (bundle) quantum evolution is described with respect to (is referred to) the corresponding classical evolution of the same system.

One can also take M to be the configuration or phase space of some observer. Then γ can naturally be defined as the observer's trajectory in the corresponding space.

At this point we want to say a few words on the possibility to identify the Hilbert bundle's base M with the phase space of certain system and to make some comments on [36], where this case is taken as a base for a bundle approach to quantum mechanics. Our generic opinion is that the phase space is not a 'suitable' candidate for a bundle's base, the reason being the Heisenberg uncertainty principle by virtue of which the points of the phase space have no physical meaning [63, chapter IV]. This reason does not apply if as a base is taken the phase space of some observer as, by definition, the observers are treated as classical objects (systems). Therefore one can set the base M of the Hilbert bundle (F, π, M) to be the phase space of some observer. Then the reference path $\gamma\colon J \to M$ can be interpreted as the observer's phase-space trajectory which, generally, can have self-intersections. The further treatment of this case is the same as of $M = \mathbb{E}^3$. Regardless of the above-said, one can always identify M with the system's phase space, if it exists, as actually M is a free parameter in the present work.

An interesting bundle approach to quantum mechanics is contained in [36]. In it the evolution of a quantum system is described in a Hilbert bundle *over the system's phase space* with the ordinary system's Hilbert space as a (typical) fibre which is, sometimes, identified with the fibre over an arbitrary fixed phase-space point. The evolution itself is presented as a parallel transport in the bundle space generated via non-dynamical linear (and symplectic) connection which is closely related to the symplectic structure of the phase space. An important feature of [36] is that in it the bundle structure is derived from the physical content of the paper. In this sense [36] can be considered as a good motivation for the general constructions in the present investigation.

Before comparing the mathematical results of [36] with the ones of this work, we have to say that the *loc. cit.* contains some incorrect 'bundle' expressions which, however, happily do not influence most of the conclusions made on their base. In [69] we point to and show possible ways for improving of a number of mathematically non-rigorous or wrong expressions, assertions, and definitions in [36]. We emphasize that all this concerns only the 'bundle' part of the mathematical structure of *loc. cit.* and does not deal with its physical contents. The general moral of the critical remarks

of [69] is: most of the final results and conclusions of [36] are valid provided the pointed in [69] (and other minor) corrections are made. Below we shall suppose that this is carefully done. On this base we will compare [36] with the present work.

The main common point between [36] and this investigation is the consistent application of the fibre bundle theory to (nonrelativistic) quantum mechanics. But the implementation of this intention is quite different: in [36] we see a description of quantum mechanics in a new, but 'frozen', geometrical background based on a non-dynamical linear connection deduced from the symplectical structure of the system's phase space, while the present work uses a 'dynamical geometry' (linear transport along paths, which may turn to be a parallel one generated by a linear connection) whose properties depend on the system's Hamiltonian, i.e. on the physical system under consideration itself.

The fact that in [36] the system's phase space is taken as a base of the used Hilbert bundle is not essential since nothing can prevent us from making the same choice as, actually, the base is not fixed here. In [36] is partially considered the dynamics of multispinor fields. This is an interesting problem, but, since it is not primary related to conventional quantum mechanics, we think it is out of the scope of our work. The methods of its solution are outlined in [36] and can easily be incorporated within our bundle quantum mechanics.

The fields of (metaplectic) spinors used in [36] are simply sections of the Hilbert bundle, while the "world-line spinors" in *loc. cit.* are sections along paths in our terminology. The family of operators $\mathcal{O}_{\phi^a}$ or $\mathcal{O}_f(\phi)$ [36, equations (4.8) and (4.9)] acting on F_ϕ are actually bundle morphisms.

A central rôle in both works plays the 'principle of invariance of the mean values' (see the paragraph after postulate 7.1): the mean values (mathematical expectations) of the liftings of paths or the morphisms along paths corresponding to the observables (dynamical variables) are independent of the way they are calculate. We have used this assumption many times (see, e.g., Sections 7, 8, and 10, in particular, equations (7.3), (8.7), (8.13), (8.39), and (10.27)). In [36] 'the invariance of the mean values' is mentioned several times and it is used practically in the form of the 'background-quantum split symmetry' principle, explained in [36, sect. 4] (see, e.g., [36, equation (4.18)] and the comments after it). Its particular realizations are written as [36, equation (4.17) and second equation (4.42)] which are equivalent to it in the corresponding context. A consequence of the mean-value invariance is the 'Abelian' character of the compatible with it connections, expressed by [36, equation (4.14)], which is a special case of our result [1, equation (4.4)]. In [36] the mean values are independent of the point at which they are determined. In our bundle quantum mechanics this is not generally the case as different points correspond to different time values (see, e.g., (7.3)). This difference clearly reflects the dynamical character of our approach and the

'frozen' geometrical one of [36].

In both works the quantum evolution is described via appropriate transport along paths: In [36, see, e.g., equations (3.54) and (4.53)] this is an 'Abelian' parallel transport along curves, whose holonomy group is $U(1)$ [36, equation (4.38)], while in our investigation is employed a transport along paths uniquely determined by the Hamiltonian (see Sect. 5) which, generally, need not to be a parallel translation.

Now we turn our attention on the bundle equations of motion: in the current work we have a *single* bundle Schrödinger equation (6.25) (see also its matrix version (6.12)), while in [36, equation (5.54)] there is an *infinite* number of such equations, one Schrödinger equation in each fibre F_ϕ for the system's state vector $|\psi(t)\rangle_\phi$ at every point $\phi \in M$.[64] Analogous is the situation with the statistical operator (compare our equation (10.27) or (10.25) with [36, equation (4.56)]). This drastical difference is due to the *different objects* used to describe systems states: for the purpose we have used *liftings of paths or sections along paths* (see Sect. 4), while in [36] are utilized *(global) sections* of the bundle defined via (4.2) (cf. [36, equation (4.41)]). Hence, what actually is done in [36] is the construction of an isomorphic images of the quantum mechanics from the fibre $\mathcal{F}$ in every fibre F_ϕ, $\phi \in M$ (see the comments after (4.2)).

To summarize the comments on part of the mathematical structures in [36]: It contains a fibre bundle description of quantum mechanics. The state vectors are replaced by (global) sections of a Hilbert bundle with the system's phase space as a base and their (bundle) evolution is governed through Abelian parallel transport arising from the symplectical structure of the phase space. Locally, in any fibre of the bundle, the evolution is presented by a Schrödinger equation, specific for each fibre of the bundle. The work contains a number of incorrect mathematical constructions which, however, can be corrected so that the final conclusions remain valid. Some ideas of the paper are near to the ones of this investigation but their implementation and development is quite different in both works.

13. Summary

In this work we have proposed and developed a new invariant fibre bundle formulation of nonrelativistic quantum mechanics. With respect to the physically predictable results, it is fully equivalent to the usual formulation of the theory in the case when the underlying manifold is the (coordinate) three dimensional Euclidean space of classical/quantum mechanics. In the new description a pure state of a quantum system is describe by a state lifting of paths or a state section along paths of a Hilbert bundle. The time

[64] Note that the appearing in [36, equations (4.54)–(4.56)] operator $\mathcal{O}_H$ is an analogue of our matrix-bundle Hamiltonian (see Sect. 6).

evolution of the state liftings/sections obeys the bundle Schrödinger equation (6.25). A mixed state of a quantum system is described via the density lifting of paths (in the bundle of point-restricted morphisms of system's Hilbert bundle) or the density morphism along paths (in system's Hilbert bundle) satisfying the Schrödinger (type) equation (10.27). In the proposed bundle approach to any dynamical variable corresponds a unique observable which is a lifting of paths (in the bundle of point-restricted morphisms of system's Hilbert bundle) or morphism along paths (in the Hilbert bundle of the investigated system). The observed value of a dynamical variable is equal (by definition) to the mean value (the mathematical expectation) of the corresponding lifting/morphism and it is calculated by means of the bundle state lifting/section or density lifting/morphism corresponding to the system's state at the moment.

The correspondence between the conventional Hilbert space description and the new Hilbert bundle description of non-relativistic quantum mechanics is given in table 1 on page 93.

A feature of the bundle form of quantum mechanics is the inherent connection between physics and geometry: the system's Hamiltonian (via equation (6.22)) completely determines the concrete properties of the system's Hilbert bundle. In this sense, here the Hamiltonian plays the same rôle as the energy-momentum tensor in general relativity. Another view-point (also based on (6.22)) is to look on the Hamiltonian as a gauge field in the sense of Yang-Mills theories. In any case, we see in the bundle quantum mechanics a realization of the intriguing idea, going back to Albert Einstein and Bernhard Riemann, that the physical properties of the systems are responsible for the geometry of the spaces used for their description.

Table 1: Comparison between Hilbert space and Hilbert bundle descriptions.

Hilbert space description	**Hilbert bundle description**	**Remark(s)**
Hilbert space $\mathcal{F}$	Hilbert bundle (F,π,M)	$l_x : F_x \to \mathcal{F}$
Vector $\varphi \in \mathcal{F}$	Section $\overline{\Phi} \in \mathrm{Sec}\,(F,\pi,M)$	$\overline{\Phi}\colon x \mapsto l_x^{-1}(\varphi)$
Operator $\mathcal{A} : \mathcal{F} \to \mathcal{F}$	Morphism $\overline{A} \in \mathrm{Mor}_M\,(F,\pi,M)$	$\overline{A}_x = l_x^{-1} \circ \mathcal{A} \circ l_x$
State vector $\psi \in \mathcal{F}$	State lifting Ψ of paths in (F,π,M)	$\Psi_\gamma(t) = l_{\gamma(t)}^{-1}(\psi(t))$
Observable $\mathcal{A}\colon \mathcal{F} \to \mathcal{F}$	Observable lifging A of paths	$A_\gamma(t) = l_{\gamma(t)}^{-1} \circ \mathcal{A}(t) \circ l_{\gamma(t)}$
Hermitian scalar product $\langle\phi|\psi\rangle$	Hermitian bundle scalar product $\langle\Phi_x|\Psi_x\rangle_x$	$\langle\cdot|\cdot\rangle_x = \langle l_x \cdot | l_x \cdot\rangle$
Hermitian conjugate operator $\mathcal{A}^\dagger$ corresponding to an operator $\mathcal{A}$: $\langle\mathcal{A}^\dagger\phi|\psi\rangle = \langle\phi|\mathcal{A}\psi\rangle$	1) Hermitian conjugate map $A^\ddagger_{x\to y}\colon F_x \to F_y$ to a bundle map $A_{y\to x} : F_y \to F_x$: $\langle A^\ddagger_{x\to y}\Phi_x|\Psi_y\rangle_y = \langle\Phi_x|A_{y\to x}\Psi_y\rangle_x$ 2) Hermitian conjugate morphism $A^\ddagger$ to a morphism A along paths: $\langle A^\ddagger_x\Phi_x|\Psi_x\rangle_x = \langle\Phi_x|A_x\Psi_x\rangle_x$	$A^\ddagger_{x\to y} = = l_y^{-1} \circ (l_x \circ A_{y\to x} \circ l_y^{-1})^\dagger \circ l_x$ $A^\ddagger_x = l_x^{-1} \circ (l_x \circ A_x \circ l_x^{-1})^\dagger \circ l_x$
Unitary operator: $\mathcal{A}^\dagger = \mathcal{A}^{-1}$	1) Unitary bundle map: $A^\ddagger_{x\to y} = A^{-1}_{y\to x} := (A_{y\to x})^{-1}$ 2) Unitary bundle morphism: $A^\ddagger = A^{-1}$	$A^{-1}_{x\to y}$ is the left inverse of $A_{x\to y}$ $A^\ddagger_x = A_x^{-1}$
Hermitian operator: $\mathcal{A}^\dagger = \mathcal{A}$	Hermitian morphism: $A^\ddagger = A$	$A^\dagger_x = A_x$; $A^\ddagger = A \iff \mathcal{A}^\dagger = \mathcal{A}$
Basis $\{f_a(t)\}$ in $\mathcal{F}$	Basis $\{e_a\}$ in $\mathrm{Sec}(F,\pi,M)$	A good choice is $e_a(x) = l_x^{-1}(f_a(t))$
Matrix corresponding to a linear map or a vector in a given basis (bases): the same notation but the kernel letter is in **boldface**		For example: $\boldsymbol{\mathcal{A}}(t)$, $\boldsymbol{A}$, $\boldsymbol{\psi}(t)$, $\boldsymbol{\Psi}_\gamma(t)$, $\boldsymbol{l}_x$, $\boldsymbol{\mathcal{U}}(t,s)$, $\boldsymbol{U}_\gamma(t,s)$
Mean value of operator $\mathcal{A}(t)$: $\langle\mathcal{A}(t)\rangle^t_\psi = \frac{\langle\psi(t)\|\mathcal{A}(t)\psi(t)\rangle}{\langle\psi(t)\|\psi(t)\rangle}$	Mean value of observable lifting $A(t)$: $\langle A(t)\rangle^t_{\Psi_\gamma} = \frac{\langle\Psi_\gamma(t)\|A(t)\Psi_\gamma(t)\rangle_{\gamma(t)}}{\langle\Psi_\gamma(t)\|\Psi_\gamma(t)\rangle_{\gamma(t)}}$	$\langle A(t)\rangle^t_{\Psi_\gamma} = \langle\mathcal{A}(t)\rangle^t_\psi$
Evolution operator $\mathcal{U}$	Evolution transport U along paths	$U_\gamma(t,s) = l_{\gamma(t)}^{-1} \circ \mathcal{U}(t,s) \circ l_{\gamma(s)}$; see (2.1) and (5.7)
1) Hamiltonian $\mathcal{H}$ 2) Matrix Hamiltonian $\boldsymbol{\mathcal{H}}^{\mathbf{m}}$	1) Bundle Hamiltonian H 2) Matrix-bundle Hamiltonian $\boldsymbol{H}^{\mathbf{m}}_\gamma$ 3) Matrix $\boldsymbol{\Gamma}$ of the coefficients of the evolution transport 4) Derivation D along paths	$H_\gamma(t) = l_{\gamma(t)}^{-1} \circ \mathcal{H}(t) \circ l_{\gamma(t)}$ See (6.11), (6.13), and (6.14) $\boldsymbol{\Gamma}_\gamma(t) = -\boldsymbol{H}^{\mathbf{m}}_\gamma(t)/\mathrm{i}\hbar$ $D^\gamma_t e_i = \Gamma^j{}_i(t;\gamma) e_j(t;\gamma)$
1) Schrödinger equation: $\mathrm{i}\hbar\frac{d\psi(t)}{dt} = \mathcal{H}(t)\psi(t)$ 2) Matrix Schrödinger equation: $\mathrm{i}\hbar\frac{d\boldsymbol{\psi}(t)}{dt} = \boldsymbol{\mathcal{H}}^{\mathbf{m}}(t)\boldsymbol{\psi}(t)$	1) Bundle Schrödinger equation: $D^\gamma_t\Psi_\gamma = 0$ or $D\Psi = 0$ 2) Matrix-bundle Schrödinger equation: $\mathrm{i}\hbar\frac{d\boldsymbol{\Psi}_\gamma(t)}{dt} = \boldsymbol{H}^{\mathbf{m}}_\gamma(t)\boldsymbol{\Psi}_\gamma(t)$	Equivalent equations
Density operator ρ	Density lifting P of paths	$\mathrm{P}_\gamma(t) = l_{\gamma(t)}^{-1} \circ \rho(t) \circ l_{\gamma(t)}$
Mean value of operator $\mathcal{A}(t)$: $\langle\mathcal{A}\rangle^t_\rho := \langle\mathcal{A}(t)\rangle^t_\rho := \mathrm{Tr}(\rho(t) \circ \mathcal{A}(t))$	Mean value of observable lifting $A(t)$: $\langle A\rangle^{t,\gamma}_{\mathrm{P}} = \langle A_\gamma(t)\rangle^t_{\mathrm{P}_\gamma} = \mathrm{Tr}(\mathrm{P}_\gamma(t) \circ A_\gamma(t))$	$\langle\mathcal{A}\rangle^t_\rho = \langle A\rangle^{t,\gamma}_{\mathrm{P}}$
Density operator evolution: $\mathrm{i}\hbar\frac{d\rho(t)}{dt} = [\mathcal{H}(t),\rho(t)]_-$	Density lifting evolution: $\widetilde{D}(\widetilde{\mathrm{P}}) = 0$ or $^\circ D(\mathrm{P}) = 0$	Equivalent equations
Schrödinger picture of motion: $\psi(t)$, $\mathcal{A}(t)$; $\rho(t)$, $\mathcal{A}(t)$	Bundle Schrödinger picture of motion: $\Psi_\gamma(t)$, $A_\gamma(t)$; $\mathrm{P}_\gamma(t)$, $A_\gamma(t)$	See: (2.6), (6.25); (10.5), (10.25), (10.27)
Heisenberg picture of motion: $\psi^{\mathrm{H}}_t(t_0)$, $\mathcal{A}^{\mathrm{H}}_t(t_0)$; $\rho^{\mathrm{H}}_t(t_0)$, $\mathcal{A}^{\mathrm{H}}_t(t_0)$	Bundle Heisenberg picture of motion: $\Psi^{\mathrm{H}}_{\gamma,t}(t_0)$, $A^{\mathrm{H}}_{\gamma,t}(t_0)$; $\mathrm{P}^{\mathrm{H}}_{\gamma,t}(t_0)$, $A^{\mathrm{H}}_{\gamma,t}(t_0)$	See: (8.14), (8.10), (8.16), (8.17); (10.28), (10.29)
'General' picture of motion: $\psi^{\mathcal{V}}_t(t_0)$, $\mathcal{A}^{\mathcal{V}}_t(t_0)$; $\rho^{\mathcal{V}}_t(t_0)$, $\mathcal{A}^{\mathcal{V}}_t(t_0)$	Bundle 'general' picture of motion: $\Psi^{V}_{\gamma,t}(t_0)$, $A^{V}_{\gamma,t}(t_0)$; $\mathrm{P}^{V}_{\gamma,t}(t_0)$, $A^{V}_{\gamma,t}(t_0)$	See: (8.47), (8.50), (8.42), (8.51), (8.52); (10.36)–(10.39)
Integral of motion $\mathcal{A}$: $\mathrm{i}\hbar\frac{\partial\mathcal{A}(t)}{\partial t} + [\mathcal{A}(t),\mathcal{H}(t)]_- = 0$	Integral of motion A: $(\widetilde{D}^\gamma_t(A))\Psi = 0$ or $^\circ D(A) = 0$	Equivalent concepts. See also: (9.9), (9.19); (9.10), (9.21)

14. Discussion

Since the set of all sections of a vector bundle is a module over the ring of all (C^0) functions on its base with values in the field with respect to which it has a vector structure [15, chapter 3, propositions 1.6], the set $\mathrm{Sec}\,(F,\pi,M)$ is a module over the ring of functions $f\colon M\to\mathbb{C}$. Besides, this module is equipped with a scalar product. In fact, if $\Phi,\Psi\in\mathrm{Sec}\,(F,\pi,M)$, and $f,g\colon M\to\mathbb{C}$, the vector structure of $\mathrm{Sec}\,(F,\pi,M)$ is given by

$$(f\Phi+g\Psi)\colon x\mapsto f(x)\Phi(x)+g(x)\Psi(x)\in F_x,\qquad x\in M \tag{14.1}$$

and its inner product is defined via $(\Phi,\Psi)\mapsto\langle\Phi|\Psi\rangle\colon M\to\mathbb{C}$ where

$$\langle\Phi|\Psi\rangle\colon x\mapsto\langle\Phi|\Psi\rangle(x):=\langle\Phi(x)|\Psi(x)\rangle_x\in\mathbb{C},\qquad x\in M. \tag{14.2}$$

Such a structure, module with an inner product, can naturally be called a *Hilbert module*.

Moreover, any morphism $A\in\mathrm{Mor}\,(F,\pi,M)$ can be considered as an operator $A\colon\mathrm{Sec}\,(F,\pi,M)\to\mathrm{Sec}\,(F,\pi,M)$ of the sections over (F,π,M) whose action is defined by

$$(A\Phi)\colon x\mapsto A_x(\Phi(x)),\qquad \Phi\in\mathrm{Sec}\,(F,\pi,M),\quad x\in M \tag{14.3}$$

and vice versa, to any operator $B\colon\mathrm{Sec}\,(F,\pi,M)\to\mathrm{Sec}\,(F,\pi,M)$ there corresponds a unique morphism of (F,π,M) whose restriction B_x on F_x is given via

$$B_x(\Phi(x)):=(B\Phi)(x),\qquad \Phi\in\mathrm{Sec}\,(F,\pi,M),\quad x\in M. \tag{14.4}$$

These mathematical results allow us, if needed, to reformulate (equivalently) the Hilbert bundle description of quantum mechanics in terms of vectors and operators, but now in the Hilbert module of sections of the Hilbert bundle over the space-time M.

Since any pure state of a quantum system can be described via a suitable density operator [9, chapter VIII, § 24], the nonrelativistic quantum mechanics is possible to be formulated entirely in terms of liftings of paths in the bundle of restricted morphisms of the Hilbert bundle of the considered system.

If unbounded states are investigated, the system's Hilbert space has to be replaced with a more general space. In our formalism, this will result in the identification of the fibre $\mathcal{F}$ with that last space. If this is the case, some problems with the infinite norms of some vectors may arise, but this is a technical task which does not change the main ideas.

In the present investigation we have not fixed the base M of the Hilbert bundle (F,π,M). We did not used even some concrete assumptions about M, except the self-understanding ones, e.g. such as that it is an non-empty

topological space. In Subsect. 4.3, we assume as a working hypothesis, M to be the (coordinate) 3-dimensional Euclidean space $\mathbb{E}^3$ of quantum (or classical) mechanics. This is required for the physical interpretation of the developed here theory. This interpretation holds true also for any differentiable manifold M with $\dim M \geq 3$. This is important in connection with further generalizations. For instance, such are the cases when M is chosen as the 4-dimensional Minkowski space-time M^4 of special relativity, or the pseudo-Riemannian space-time V_4 of general relativity, or the Riemann-Cartan space-time U_4 of the U_4-gravitational theory. All this points to the great arbitrariness in the choice of the geometrical structure of M. Generally it has to be determined by a theory different from quantum mechanics, such as the classical mechanics or the special or general relativity. As a consequence of this, there is a room for some kind of unified theories, for instance for a unification of quantum mechanics and general relativity. These problems will be investigated elsewhere.

The developed in the present investigation theory is global in a sense that in it we interpret M as being the whole space(-time) where the studied objects 'live'. If by some reasons one wants or is forced to consider a system resided into a limited region of M, then all the theory can be localized by replacing M with this region or simply via *mutatis mutandis* restricting the already obtained results on it. At the same time, our theory is local in a sense that such are the used in it observers which are assumed to can make measurements only at the points they reside. The theory can slightly be generalized by admitting that the observers can perform observations (measurements) at points different from their own residence. This puts the problem of defining the mean values of the observables at points different from the one at which the observer is situated in such a way as they to be independent of the possibly introduce for this purpose additional constrictions (cf. [1, sect. 3]). This problem will be solved elsewhere.

The observers we have been dealing until now in this work can be called scalar and point-like as it is supposed that they have no internal structure and the only their characteristics in the theory are their positions (generally) in the space-time M. Notice that as a set the manifold M coincides with the variety of all possible positions of all possible observers. This observation suggests that in the general case M has to be replaced with a set $\widetilde{M}$ consisting of all values of all parameters describing completely the states of every possible observer. Naturally, the set $\widetilde{M}$ has to be endowed with some topological or/and smooth geometrical structure. For instance, consider an 'anisotropic' point-like observers characterized by their position x in M and some vector V_x at it, i.e. by pairs like (x, V_x), $x \in M$, $V_x \in L_x$ with L_x being some vector space. In this case $\widetilde{M}$ is naturally identified with the (total) bundle space L of the arising over M vector bundle (L, π_L, M) with $L := \bigcup_{x\in M} \bigcup_{V_x\in L_x} (x, V_x)$ and $\pi_L(x, V_x) := x \in M$, i.e. we can put $\widetilde{M} = L$.

In particular, if V_x is the observer's velocity at x, we have $\widetilde{M} = T(M)$ with $(T(M), \pi_{T(M)}, M)$ being the bundle tangent to M. Another sensible example is when V_x is interpreted as observer's spin etc. In connection with some recent investigations (see, e.g., [21, 36]) it is worth to be studied the special case when $\widetilde{M}$ is taken to be the (classical) phase space of the observer. Returning to the general situation, we see that our theory can easily be modified to cover such generalizations. For this purposed we have simply to replace the Hilbert bundle (F, π, M) over M with the Hilbert bundle $(F, \pi, \widetilde{M})$ over $\widetilde{M}$. This, together with other evident corresponding changes, such as $(x \in M) \mapsto (x \in \widetilde{M})$ and $(\gamma\colon J \to M) \mapsto (\gamma\colon J \to \widetilde{M})$, allows us to apply the developed in the present investigation Hilbert bundle description to far more general situations than the one we were speaking about until now. In principle, the afore-described procedure is applicable to non-local observers too, but this is out of the subject of the present work.

We want also to mention that since any fibre of the Hilbert bundle (F, π, M) is an isomorphic image of the Hilbert space $\mathcal{F}$, the conventional probabilistic interpretation of the nonrelativistic quantum mechanics [8, 9] remains *mutatis mutandis* completely valid in the Hilbert bundle description too. For this purpose one has to replace state vectors and operators acting on $\mathcal{F}$ with the corresponding state liftings of paths and liftings of paths in (F, π, M) and then to follow the general rules outlined in this work (see the paragraph after remark 4.2) and, for instance, in [9].

In connection with further applications of the bundle approach to the quantum field theory, we notice the following. Since in this theory the matter fields are represented by operators acting on (wave) functions from some space, the matter fields in their bundle modification should be described via morphisms along paths of a suitable fibre bundle whose sections (along paths) will represent the wave functions. We can also, equivalently, say that in this way the matter fields would be sections of the fibre bundle of bundle morphisms of the mentioned suitable bundle. An important point here is that the matter fields are primary related to the bundle arising over the space-time (or other space which includes it) and not to the space-time itself to which other structures are directly related, such as connections and the principle bundle over it.

The bundle approach to nonrelativistic quantum mechanics, developed in the present investigation, seems also applicable to classical mechanics, statistical mechanics, relativistic quantum mechanics, and field theory. We hope that such a novel treatment of these theories will reveal new perspectives for different generalizations, in particular for their unification with the theory of gravitation.

References

[1] Bozhidar Z. Iliev. Quantum mechanics from a geometric-observer's viewpoint. *Journal of Physics A: Mathematical and General*, 31(4):1297–1305, January 1998. (LANL xxx archive server, E-print No. quant-ph/9803083).

[2] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. I. Introduction. The evolution transport. (LANL xxx archive server, E-print No. quant-ph/9803084), 1998.

[3] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. II. Equations of motion and observables. (LANL xxx archive server, E-print No. quant-ph/9804062), 1998.

[4] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. III. Pictures and integrals of motion. (LANL xxx archive server, E-print No. quant-ph/9806046), 1998.

[5] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. IV. Mixed states and evolution transport's curvature. (LANL xxx archive server, E-print No. quant-ph/9901039), 1999.

[6] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. V. Theory's interpretation, summary, and discussion. (LANL xxx archive server, E-print No. quant-ph/9902068), 1999.

[7] P. A. M. Dirac. *The principles of quantum mechanics*. Oxford at the Clarendon Press, Oxford, fourth edition, 1958.

[8] V. A. Fock. *Fundamentals of quantum mechanics*. Mir Publishers, Moscow, 1978.

[9] A. M. L. Messiah. *Quantum mechanics*. Interscience, New York, 1958. (Vol. I and vol. II.).

[10] E. Prugovečki. *Quantum mechanics in Hilbert space*, volume 92 of *Pure and applied mathematics*. Academic Press, New York-London, second edition, 1981.

[11] L. D. Landau and E. M. Lifshitz. *Quantum mechanics: non-relativistic theory*, volume III of *Course of theoretical physics*. Pergamon Press, London, 2 edition, 1965. (Translation from Russian, Nauka, Moscow, 1962, 1974).

[12] Bernard F. Schutz. *Geometrical methods of mathematical physics*. Cambridge University Press, Cambridge-London-New York- New Rochelle-Melbourne-Sydney, 1982. (Russian translation: Mir, Moscow, 1984).

[13] R. Coquereaux. *Riemannian geometry, fibre bundles, Kaluza-Klein theories and all that*, volume 16 of *World Scientific Lecture Notes in Physics*. World Scientific, Singapore-New York-Hong Kong, 1988.

[14] N. P. Konopleva and V. N. Popov. *Gauge fields*. Hardwood Academic Publishers, Chur-London-New York, second edition, 1981. (Original Russian edition: Atomizdat, Moscow, 1972 (1 ed.), 1980 (2 ed.)).

[15] D. Husemoller. *Fibre bundles*. McGraw-Hill Book Co., New York-St. Louis-San Francisco-Toronto-London-Sydney, 1966.

[16] R. Hermann. *Geometry, physics, and systems*. Marcel Dekker, Inc., New York, 1973.

[17] R. Hermann. *Vector bundles in mathematical physics*, volume I. W. A. Benjamin, Inc., New York, 1970.

[18] R. Hermann. *Vector bundles in mathematical physics*, volume II. W. A. Benjamin, New York, 1970.

[19] T. W. B. Kibble. Geometrization of quantum mechanics. *Commun. Math. Phys.*, 65(2):189–201, 1979.

[20] J. Anandan. A geometric view of quantum mechanics. In Anandan J. S., editor, *Quantum coherence*, Proceedings of the Conference on Fundamental Aspects of Quantum theory, Columbia, South Carolina, 14–16 December 1989. World Scientific, Singapore, 1990. (See also Preprint MPI-PAE/PTh 77/90, 1990 and Found. Phys., 21(11):1265–1284, 1991).

[21] A. Ashtekar and T. A. Schilling. Geometrical formulation of quantum mechanics. LANL xxx archive server, E-print No. gr-qc/9706069, 1997. (See also preprint CGPG 97/6-1).

[22] D. C. Brody and L. P. Hughston. Geometrization of statistical mechanics. LANL xxx archive server, E-print No. gr-qc/9708032, 1997.

[23] J. Anandan and Y. Aharonov. Geometry of quantum evolution. *Phys. Rev. Lett.*, 65(14):1697–1700, 1990.

[24] A. Uhlmann. A gauge field governing parallel transport along mixed states. *Lett. Math. Phys.*, 21(3):229–236, March 1991.

[25] F. Wilczek and A. Zee. Appearance of gauge structure in simple dynamical systems. *Phys. Rev. Lett.*, 52(24):2111–2114, 1984.

[26] A. Uhlmann. Parallel transport of phases. In Henning J. D., Lücke W., and Tolar J., editors, *Differential geometry, Group representations, and*

Quantization, number 379 in Lecture Notes in Physics, pages 55–72. Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[27] H. J. Bernstein and A. V. Phillips. Fibre bundles and quantum theory. *Scientific America*, 245:94–104, July 1981. (Reprinted: Uspekhi Fizicheskikh Nauk, vol. 136, No. 4, pp. 665–692, 1982 (In Russian)).

[28] A. Echeverria-Enriquez, M.C. Munoz-Lecanda, N. Roman-Roy, and C. Victoria-Monge. Mathematical foundations of geometric quantization. *Extracta Math.*, 13:135–238, 1998. LANL xxx archive server, E-print No. math-ph/9904008, 1999.

[29] W. Drechsler and Ph. A. Tuckey. On quantum and parallel transport in a Hilbert bundle over space-time. *Classical and Quantum Gravity*, 13(4):611–632, April 1996.

[30] J. Coleman. On parallel transport in quantum bundles over Robertson-Walker spacetime. LANL xxx archive server, E-print No. gr-qc/9605067, 1996.

[31] Bozhidar Z. Iliev. Normal frames and linear transports along paths in vector bundles. LANL xxx archive server, E-print No. gr-qc/9809084, 1998.

[32] Bozhidar Z. Iliev. Linear transports along paths in vector bundles. I. General theory. JINR Communication E5-93-239, Dubna, 1993.

[33] D. Graudenz. On the space-time geometry of quantum systems. Preprint CERN-TH.7516/94, CERN, November 1994. (See also LANL xxx archive server, E-print No. gr-qc/9412013).

[34] D. Graudenz. The quantum gauge principle. Preprint CERN-TH/96-107, CERN, April 1996. (See also LANL xxx archive server, E-print No. hep-th/9604180).

[35] M. Asorey, J. F. Cariñena, and M. Paramio. Quantum evolution as a parallel transport. *Journal of Mathematical Physics*, 21(8):1451–1458, 1982.

[36] Martin Reuter. Quantum mechanics as a gauge theory of metaplectic spinor fields. *Int. J. Mod. Phys.*, A13:3835–3884, 1998. (See also: preprint DESY 97-127, 1997 and LANL xxx archive server, E-print No. hep-th/9804036, 1998).

[37] Bozhidar Z. Iliev. Transports along paths in fibre bundles. General theory. JINR Communication E5-93-299, Dubna, 1993.

[38] J. von Neumann. *Mathematical foundations of quantum mechanics*. Princeton Univ. Press, Princeton, New Jersey, 1955.

[39] M. Reed and B. Simon. *Methods of modern mathematical physics*, volume I, II, III, IV. Academic Press, New York, 1972, 1975, 1979, 1978.

[40] Willy-Hance Steeb. *Hilbert spaces, wavelets, generalized functions and modern quantum mechanics*, volume 451 of *Mathematics and its applications*. Kluwer Acad. Publ., Dordrecht-Boston-London, 1998.

[41] Serge Lang. *Differential manifolds*. Springer Verlag, New York, 1985. (originally published: Addison-Wesley Pub. Co, Reading, Mass. 1972).

[42] J. D. Bjorken and S. D. Drell. *Relativistic quantum mechanics*. McGraw-Hill Book Company, New York, 1964.

[43] N. Steenrod. *The topology of fibre bundles*. Princeton Univ. Press, Princeton, ninth edition, 1974.

[44] W. Greub, S. Halperin, and R. Vanstone. *De Rham cohomology of manifolds and vector bundles*, volume 1 of *Connections, Curvature, and Cohomology*. Academic Press, New York and London, 1972.

[45] Ph. Hartman. *Ordinary Differential Equations*. John Wiley & Sons, New York-London-Sydney, 1964.

[46] Sze-Tsen Hu. *Homotopy Theory*. Academic Press, New York-London, 1959.

[47] Bozhidar Z. Iliev. Transports along paths in fibre bundles. II. Ties with the theory of connections and parallel transports. JINR Communication E5-94-16, Dubna, 1994.

[48] Bozhidar Z. Iliev. Parallel transports in tensor spaces generated by derivations of tensor algebras. JINR Communication E5-93-1, Dubna, 1993.

[49] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*, volume I. Interscience Publishers, New York-London, 1963.

[50] J. A. Schouten. *Ricci-Calculus: An Introduction to Tensor Analysis and its Geometrical Applications*. Springer Verlag, Berlin-Göttingen-Heidelberg, second edition, 1954.

[51] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge Univ. Press, Cambridge, 1973.

[52] J. L. Synge. *Relativity: The General Theory*. North-Holland Publ. Co., Amsterdam, 1960.

[53] B. B. Walwadkar. Truesdell transport in general relativity. *General Relativity and Gravitation*, 15(12):1107–1116, 1983.

[54] B. B. Walwadkar and K. V. Virkar. Truesdell invariance in relativistic electromagnetic fields. *General Relativity and Gravitation*, 16(1):1–7, 1984.

[55] L. Radhakrishna, L. N. Katkar, and T. H. Date. Jaumann transport in relativistic continuum mechanics. *General Relativity and Gravitation*, 13(10):939–946, 1981.

[56] R. Dandoloff and W. J. Zakrzewski. Parallel transport along a space curve and related phases. *J. Phys. A: Math. Gen.*, 22:L461–L466, 1989.

[57] Bozhidar Z. Iliev. Transports along maps in fibre bundles. JINR Communication E5-97-2, Dubna, 1997. (LANL xxx archive server, E-print No. dg-ga/9709016).

[58] James D. Stasheff. "Parallel" transport in fibre spaces. Reimpreso del boletin de la sociedad matematica mexicana, Sociedad Matematica Mexicana, Printed in USA, 1966. pages 68–84.

[59] Bozhidar Z. Iliev. Transports along paths in fibre bundles. III. Consistency with bundle morphisms. JINR Communication E5-94-41, Dubna, 1994. (LANL xxx archive server, E-print No. dg-ga/9704004).

[60] Peter Dombrowski. Krümmungsgrößen gleichungsdefinierter Untermannigfaltigkeiten Riemannscher Mannigfaltigkeiten. *Mathematische Nachrichten*, 38(3/4):133–180, 1968.

[61] Richard G. Cooke. *Infinite matrices and sequence spaces*. London, 1950. (Reprinted: Dover Publications, New York, 1966).

[62] Bozhidar Z. Iliev. Linear transports along paths in vector bundles. II. Some applications. JINR Communication E5-93-260, Dubna, 1993.

[63] A. M. L. Messiah. *Quantum mechanics*, volume I. North Holland, Amsterdam, 1961.

[64] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[65] K. Rostquist and M. Goliath. Lax pair tensors and integrable spacetimes. LANL xxx archive server, E-print No. gr-qc/9707003, 1997.

[66] Bozhidar Z. Iliev. Linear transports along paths in vector bundles. III. Curvature and torsion. JINR Communication E5-93-261, Dubna, 1993.

[67] Bozhidar Z. Iliev. Linear transports along paths in vector bundles. V. Properties of curvature and torsion. JINR Communication E5-97-1, Dubna, 1997. (LANL xxx archive server, E-print No. dg-ga/9709017).

[68] V. Aldaya and J. A. de Azcárraga. Symmetries of the schrödinger bundle. *J. Math. Phys.*, 24(7):1425–1427, July 1981.

[69] Bozhidar Z. Iliev. Comments on: "Quantum mechanics as a gauge theory of metaplectic spinor fields" by M. Reuter [Int. J. Mod. Phys., vol. A13 (1998), 3835–3884; LANL E-print No. hep-th/9804036]. *International Journal of Modern Physics A*, 14(26):4153–4159, 1999. (LANL xxx archive server, E-print No. quant-ph/9902067).